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PHYSICAL REVIEW.

A DETERMINATION OF THE NATURE OF THE ELECTRIC AND MAGNETIC QUANTITIES AND OF THE DENSITY AND ELASTICITY OF THE ETHER. I.

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PART I.

INTRODUCTION.

THE advance of science is not a continuous sequence of successes, with, day by day, verifying experiment hot following brilliant theory and completer theory advented by suggestive experiment. Here and there in the field are positions where the pause

seems absolute and where the expert of a single tool, after a lifetime of unsuccessful toil, may dejectedly believe the years to have passed in profitless endeavor.

Yet the work is not wasted, it is only that what was looked for to appear as kinetic is stored as potential; in due time this will take the desired form, and add mightily to the velocity of the advance. The forging of new weapons of attack, the running of parallels, the exact determination of the difficulties to be overcome, are a part of the advance as much as is the actual motion over the field, and in the end cause no delay.

When Maxwell wrote the last word of his electromagnetic theory he had reached a point beyond which we have yet to pass. His theory has been confirmed by the experiments of Rowland, Lodge and Hertz; it has been broadened and generalized by the masterful and masterly labors of Heavyside; it has been elucidated by the acute and penetrating analysis of Poincaré; Helmholtz, Fitzgerald and other of the great physicists have aided in its development, and yet we have made no marked advance as regards our knowledge of the nature of the electric and magnetic quantities.

And this is the more remarkable when we consider the great number of striking discoveries which have been made since then. Even in the direction considered by Faraday as perhaps the most promising point of attack, that is the behavior of vacuum tubes, the magnificent results obtained by Crookes, Thomson, Lenard and Röntgen are as yet not capable of being utilized for our purpose.

There has been no lack of growth of increase of mathematical power. Our knowledge of vortex motion, for instance, has been brought, by the papers of Thomson, Kelvin, Fitzgerald and Hicks, to a point which but a few years ago could hardly have been hoped for. In addition, the genius of Heavyside has placed at our disposal an engine of surpassing strength for physical work, though perhaps at present there will be more of us to respectfully admire it than to attempt to use it.

It will be of interest to briefly review the present status.

We can conceive of the action of one quantity of electricity upon another quantity of electricity, of one quantity of magnetism upon another quantity of magnetism, of a moving quantity of electricity

upon a quantity of magnetism, and of the motion of an electro-magnetic wave, as taking place in the free ether, in space entirely devoid of matter. Under these circumstances we have no lack of perfectly satisfactory theories. The emission theory, the elastic solid theory, the rotational-elasticity theory, all these give complete solutions. It is only when we come to consider the effect of matter that we meet with any difficulty.

Corresponding to the particular theory which we may choose to adopt, we have to assume a theory of the constitution of matter, and no theory has as yet been proposed which does not break down at one or more points.

The theories may be divided into two great classes, those which make the virtual density of the ether the same in all media, which is equivalent to making the magnetic permeability the density of the ether, and those which make the virtual elasticity of the ether the same in all media, which is equivalent to making the specific inductive capacity the density of the ether.

In the first of these classes are the theories of Fresnel, Kelvin and Fitzgerald.[1] In the second class are the theories of Neumann, MacCullagh and Larmor. Several theories of each class have been treated of with great power by Heavyside in his Electro-Magnetic Theory.

The point where it is generally conceded the theories of the first type break down, is in satisfying the conditions of continuity at the interface of different media.[2]

The second type of theories escapes, as Larmor has shown, from this difficulty. But Heavyside has pointed out, in the case of some of the theories of this class, that a new difficulty arises, in that the form of the equation expressing the loss of energy in a conductor due to its resistance is incompatible with experiment. And this defect, it seems to me, must exist in all theories of this class, including the one so admirably developed by Larmor.

[1] On a Hydro-Dynamic Hypothesis as to Electro-Magnetic Action, Proc. Roy. Dublin Soc., March 25, 1899.

[2] In this connection it may be asked if in the case of iron the outermost layers of atoms are magnetic (Ewing's experiments on joints seem to show that they are not), and if it may not possibly be found that there is an absorption of light on reflection from a magnetized pole piece which is a function of the rotation impressed.

There is thus apparently nothing to choose between the two classes of theories. Each class seems capable, by proper development where this has not already been done, of expressing practically all of our experimental facts, and each breaks down at one single point.[1]

There is no obvious reason why the theories should have been restricted to these two classes, for whilst, as Maxwell pointed out, the ether certainly has inertia, it does not follow that either permeability or capacity is its density. The fact remains, however, that theories of these two types are the only ones which have given even approximately satisfactory results, and it will later be shown why this must be so.

The problem divides itself into two parts, the nature of the quantities involved, and their configuration. Of both of these we are entirely ignorant. A knowledge of the former means a full solution of the problem of the nature of electricity and magnetism, in the same sense, in which we say that we know the nature of sound when we have discovered that it is produced by waves in a medium.[2]

Knowing the nature of the electric and magnetic quantities, there is a choice of suitable configurations, any one of which will be entirely satisfactory. Since, however, the true configuration must also account for gravitation and inertia, it would be futile to develop the theory to any extent in connection with a configuration which did not at the same time give promise of results in this direction also.

In the present paper the nature of the electric and magnetic quantities will be determined, and a configuration will be indicated which may serve us for the present.[3]

[1] It is possible that each of the perfected theories of each type may represent an actual material universe, the two universes possibly interpenetrating each other. We, for the present at least, are concerned with but one, and which one has not yet been shown.

[2] We have of course no idea what a wave is; we figure it as motion in a certain way, but the ultimate motion may be something entirely different; it may, for instance, be some complicated form of rotary motion.

[3] The determination of a configuration which will explain, in addition, the nature of gravity, must be left to the future, as there appears to be at least two possible solutions and an additional independent physical relation must be discovered before we can say with certainty which is correct.

It will be shown that the number of electric and magnetic relations which can be expressed in terms of the electric, magnetic and mechanical quantities,[1] is not sufficient for a solution. There are four unknown quantities and but three independent phenomena. Hence some new relation must first be discovered.

This, it will be shown, is a relation between the magnetizing force, H and the magnetization, I, expressed by the formula

$$\frac{dI}{dH} = c\frac{I^2}{H^2}$$

or, expressed in words, "For all magnetic substances, the rate of variation of magnetization with magnetizing force is proportional to the square of the magnetic susceptibility."[2]

It is shown that this leads to a solution, which is confirmed by additional evidence in the shape of other new relations. From one of the latter, data are obtained from which a determination of the density and elasticity of the ether is made.

It thus happens that, exactly a century after the discovery of Volta, we are permitted a knowledge of the nature of the electric current.[3]

Dimensional Formulæ and Qualitative Mathematics.

Fourier's statement[4] with respect to dimensions is: "It must now be remarked that every undetermined magnitude or constant has one dimension proper to itself and that the terms of one and the same equation could not be compared if they had not the same exponent of dimensions. This consideration * * * is derived from primary notions on quantities, for which reason, in geometry and mechanics, and is the equivalent of the fundamental lemmæ which the Greeks have left us without proof."

[1] By mechanical quantities is meant quantities which are known functions of length, mass, and time, only.

[2] Or translated into the elastic problem, it behaves like rubber. See Molecular Physics, Franklin Inst. Proc., Feb., 1897.

[3] This paper was completed in July, 1898, but has been held for consideration until now. It originally contained a section on the methods of employing qualitative mathematics for the discovery of new phenomena, and the working of physical problems. On consideration this has been omitted as not being in a sufficiently developed state.

[4] Theory of Heat: Freeman's translation, p. 128.

Dimensional formulæ were used extensively by Maxwell, who says :[1] "When a given unit varies as the nth power of these (three fundamental) units, it is said to be of the nth dimensions as regards that unit." "In all dynamical sciences it is possible to define these units (of different kinds of quantities to be measured) in terms of the three fundamental units of length, mass and time."

In several instances, however, for example in treating of viscosity,[2] he suppresses one of the dimensions. In the case instanced he finds, after so doing, that the dimensions of viscosity, measured in the kinematic way, are L^2/T, whilst he has previously shown that measured according to the dynamic method its dimensions are M/LT. The former dimensions are arrived at by taking the unit of mass as the mass of unit volume of the substance considered.

This suppression of one or more of the fundamental dimensions in the formula for a quantity seems to me to be, whilst no doubt perfectly legitimate, (since we have a right to make what rules we find convenient, provided we are consistent in our use of them), a considerable departure from what I take to be Fourier's idea. For if we suppress dimensions, then, instead of Fourier's "one dimension proper to itself," a physical quantity may have an infinite number of dimensional formulæ, the one we use depending upon which units we choose and which of those chosen we suppress.

We find numerous instances of this dropping of dimensions elsewhere. For example, in Everett's book on C. G. S. units we have the following equation,[3]

$$M = L^3/T^2.$$

Whatever Fourier's conception really was, at the present time dimensions are generally treated as being merely an affair of units, any particular dimensions appearing or disappearing as we vary the units with which we work. It is evident that such a system, in which the formula for any particular quantity may be varied at pleasure, can be of little use as a method of mathematical investigation, and some of our most able mathematical physicists have expressed an opinion to that effect.

[1] Elect. and Mag., Vol. I., p. 2.
[2] Theory of Heat, p. 299.
[3] C. G. S. System of Units, p. 73.

J. J. Thomson says,[1] "It may be well to state at the outset that the dimensions of the electrical quantities are a matter of definition and depend entirely upon the system which we adopt." Also, "A theory of dimensions cannot tell us what electricity is; its object is merely to enable us to find the change in the numerical measure of a given charge of electricity or any other electrical quantity, when the units of length, mass and time are changed in any determinate way."

Webster is equally definite. To quote[2]: "The absolute dimensions of either factor are arbitrary. Attempts have been made to settle the real dimensions of κ and μ, but they are evidently based upon misconceptions of the theory of dimensions."

The weight of authority is very greatly in support of the definition of dimensions as merely arbitrary formulæ, expressing the relation between certain quantities and the units we chose to work with.

There is, however, a branch of mathematics of great power and capable of extensive and important application in physics, more especially as a means of discovering new phenomena, in which the formula assigned to any quantity represents its ultimate nature, whatever that may be. Of course, this does not imply that we *know* what its ultimate nature is, but this does not invalidate results obtained by its means. ML^2 is motion of inertia, no matter what M is ultimately, and nothing which we may find out about mass can change this fact. M may contain three or four factors, but, still, the product of these factors, which we represent by M, will when multiplied by L^2 continue to represent moment of inertia.

In view of the opinions and definitions of the eminent naturalists quoted above as to the meaning of the term dimensions it would be improper for me to apply that term in any other sense. I shall therefore call the formula expressing the ultimate nature of a quantity its "quality," and this branch of mathematics "qualitative mathematics." This term seems suitable, for the method is used to make deductions with reference to the *qualities* of a physical entity, as distinguished from other branches which concern themselves with

[1] Elements of Elect. and Mag., p. 447.
[2] Theory of Elect. and Mag., p. 418.

quantity or direction. The term has a precedent in science, for we speak of quantitative and qualitative chemical analysis.

The difference between the dimensional formula and the qualitative formula or quality of a thing is that, according to the definition of the writers quoted above, the *dimensions* "are arbitrary," are "merely a matter of definition and depend entirely upon the system of units we adopt," whilst the *quality* is an expression of the absolute nature and never varies, no matter what system of units we adopt. For this to be true, no qualities must be suppressed.[1]

All the phenomena of physics with which we are acquainted deal with the following quantities only :[2]

Name.	Symbol.
1. Quantity of gravity,	G
2. Quantity of electricity,	Q
3. Quantity of magnetism,	P
4. Gravitational coefficient,	γ
5. Specific inductive capacity,	κ
6. Magnetic permeability,	μ
7. Mass,	M
8. Length,	L
9. Time,	T

The defining equations being

$$\frac{G^2}{\gamma} = \frac{Q^2}{\kappa} = \frac{P^2}{\mu} = \frac{ML^3}{T^2}.$$

Any physical phenomena as, for instance the vibration of a stretched string, may be concerned with all these quantities, for it may be so long and heavy that gravitational force may be appreciable, it may have an electric charge on it, and it may be magnetized and vibrating in a medium of a given permeability and specific

[1] As an example of the difference between dimensional and qualitative formula we may take the result above referred to as contained in Everett's C. G. S. Units, p. 73. It is there shown that by taking the unit of mass as that which produces unit acceleration at unit distance, the dimensions of mass become $M = L^3/T^2$. It is evident that so soon as we attempt to use these dimensions in other than gravitational problems we must get incorrect results owing to the limitations we have placed upon ourselves. In the qualitative system, however, M is always M, if we take M as a fundamental unit, and if we take any other unit M is always represented by terms which are equal to M, so that no matter what branch of physics we work in we always use the same formula.

[2] This statement is as incapable of proof, and for the same reasons, as the statement embodying the law of the conservation of energy.

inductive capacity. If these quantities were not all expressible in terms of one or more of three fundamental quantities we should need, therefore, no less than six distinct experiments to completely define each phenomenon. As it is, however, we need but three. From this also it follows that, if the phenomenon can exist whilst concerning but three of the above eight quantities, the effect of the introduction of the other quantities is merely to add terms of the same quality as the original one.[1]

As an example, let us take a bead on a string. Its time of vibration depends upon, amongst other things, its mass. If it is charged the effective mass is increased, as is shown by J. J. Thomson (Application of Dynamics to Physics and Chemistry, p. 32), by an amount $2/15(\mu Q^2/a)$ (Heavyside says $\frac{1}{3}$), where Q is the charge, a the radius of the sphere, and μ the permeability of the medium. Here $\mu Q^2/a$ has the dimensions M. If the pendulum had a magnetic charge, there would be a term $P^2\varkappa/a$, and if gravity had also to be taken into account we should have a term $G^2\varkappa\mu/\gamma a$.

[1] A symmetrical system might also be made by introducing coefficients analogous to κ and μ into the equations connecting M and G, as was done by the writer (Elect. World, May 18, 1895). Also (as was done, *ibid.*) by introducing Ψ, the symbol for energy, and making it the third fundamental unit with L and T. For the present the above will be used.

In addition to these, however, we have terms which express configuration and terms which denote direction or operations. In brief, this branch of mathematics may be taken as a very precise logic.

As an example of a term depending upon configuration we may take the case of temperature. The quality of temperature, as may readily be shown, is

$$\theta = ML^2/T^2\mathrm{o},$$

where θ represents the quality of temperature and o the quality of an atom, temperature being the amount of kinetic energy per atom. It will thus be seen that whenever we have a qualitative equation in which some physical quality is shown to depend upon temperature, that quality must also be a function of the atomic weight, atomic volume, atomic valency or atomic magnetic moment.

Here the term atom is defined as that configuration upon which depends the various properties which distinguish the atom from the surrounding ether. By configuration is meant not only the relative positions, but also the relative lengths and velocities of the different parts of the atom. When we know these, however, we can split o up into separate terms. Even at present we can do this to some extent. For instance, from Faradays's law of electrolysis we have $M/$ atomic weight $= Q/$ valency. From which we get atomic weight $\div T =$ valency, which is interesting in view of Hicks's work.

Previous Work.

Maxwell, Kelvin and other writers have given dimensional formulæ for the electric and magnetic quantities, in which the dimensions of $\varkappa$ and μ were suppressed. So far as I am aware the complete dimensions or qualities in terms of L, M, T, $\varkappa$ and μ were first given by Rücker, in the *Philosophical Magazine*, February, 1889.[1]

It is also pointed out in the above-mentioned paper that if the electric and magnetic quantities are of the same nature as those met with in mechanics, then, since the ratio of the dimensions in terms of $\varkappa$ to those in terms of μ is a power of $\varkappa^{\frac{1}{2}}$, $\mu^{\frac{1}{2}}$, L and T and the two expressions must be identical, $\varkappa^{\frac{1}{2}}\,\mu^{\frac{1}{2}}$ must equal T/L, which is Maxwell's equation.

Lodge has given, in his Modern Views of Electricity, p. 702, 1889, expressions for the electric and magnetic quantities in terms of L, M, T, $\varkappa$ and μ, and a further list, taking μ to be a density. He points out that this is an assumption, and refers to Fitzgerald's suggestion that both $\varkappa$ and μ may have the same dimensions.

Williams (*Phil. Mag.*, Sept., 1892) published a very valuable paper, in which, making certain very natural assumptions, he arrived at the result that there are but two possible solutions, either μ is a density and $\varkappa$ a compliancy, or vice versa.

The writer (*Electrical World*, May 18, 1895) showed that all the electrical and magnetic relations which had been exactly determined could be brought under three equations. From this it was deduced that, since there are four quantities concerned, an infinite number of theories was possible, all of which would be in agreement with the above mentioned relations. It was also pointed out that further experiments must be made to obtain a fourth equation.

Joubin (*Journal de Phys.*, Sept., 1896) came to the conclusion from an examination of the manner in which the exponents of L, M, T, $\varkappa$ and μ vary when the electrical and magnetic quantities are written in terms of them, that μ must be a density and $\varkappa$ a compliancy.

There are considerable gaps in the reasoning by which this result is reached. After reducing the number of possible theories to

[1] Though the fact that $\varkappa$ and μ should be included in the dimensions of these quantities had previously been pointed out, as I learn from Professor Rücker, by other physicists.

twelve, the author remarks that they are all impossible with the exception of the one which makes μ a density. No reason is given for this statement, which moreover is incorrect.

The writer (*Electrical World*, Jan. 2, 1897) pointed out that the most promising field of search for the fourth equation was amongst the following :

1. Hall's effect. 2. Kerr's electrostatic effects. 3. Fizeau's experiment. 4. Magnetic rotation of light and other outlying phenomena.

It was also pointed out that our knowledge of these phenomena is not at present sufficient to enable us to use them to obtain the desired equation.

Joubin (*Journal de Phys.*, Feb., 1897), following the method described in the paper just referred to, though apparently without knowledge of it, or of the warning given as to the risk of applying the method without further experiments, by considering certain formulæ which he takes as representing the phenomena of thermo-electricity, magnetic rotation of light and piezo-electricity, arrives at the conclusion that the result reached in his former paper is correct. There is however, little justification for the formulæ adopted to represent the phenomena considered, and with very little change the arguments advanced might be used to prove that it is $\varkappa$ which is the density.

Abstracts from Some Earlier Papers.

In the Electrical World, May 18, 1895, the writer pointed out that all the dynamical relations which have been accurately formulated connecting the electric and magnetic quantities are contained in the first three of the following equations :

(1) $$Q/P = \varkappa^{\frac{1}{2}}/\mu^{\frac{1}{2}},$$

(2) $$QP = ML^2/T,$$

(3) $$\varkappa^{\frac{1}{2}}\mu^{\frac{1}{2}} = T/L,$$

(4) $$\varkappa^{\frac{1}{2}}/\mu^{\frac{1}{2}} = Z.$$

The first equation is obtained from Coulomb's law,

$$QQ'/L^2\varkappa = ML/T^2$$

and the corresponding magnetic formula

$$PP'/L^2\mu = ML/T^2.$$

The second equation is from Faraday's law, that the time rate of change of induction P/T, is a measure of the voltage, voltage multiplied by current giving power, *i. e.*,

$$P/T \times Q/T = ML^2/T^3.$$

The third equation is obtained from Weber's law, that a circuit carrying a current is equivalent in its magnetic effect to a magnetic shell whose moment is equal to the area enclosed by the contour of the circuit, *i. e.*,

$$Q/T \times \mu \times L^2 = PL,$$

by substituting for Q/P its value in terms of $\varkappa$ and μ, as given from the first equation.

This third equation is known as Maxwell's equation.

It will be seen that we have four unknown quantities, Q, P, $\varkappa$ and μ, whilst we have but three equations. Put $\varkappa^{\frac{1}{2}}/\mu^{\frac{1}{2}} = Z$ for a fourth equation, and solve—we get,

$$(5) \qquad Q = M^{\frac{1}{2}}LZ^{\frac{1}{2}}/T^{\frac{1}{2}},$$

$$(6) \qquad P = M^{\frac{1}{2}}L/T^{\frac{1}{2}}Z^{\frac{1}{2}},$$

$$(7) \qquad \varkappa = TZ/L,$$

$$(8) \qquad \mu = T/LZ.$$

By giving different values to Z we obtain all possible theories of electricity, *i. e.*, all theories which do not conflict with the known laws of electricity and magnetism.[1]

For example, let $Z = L^2T/M$.

On substituting we get

$$Q = L^2 \qquad F = M/LT^2$$

$$P = M/T \qquad H = L/T$$

$$\varkappa = LT^2/M \qquad \mu = M/L^3$$

where F is the drop in electrical potential per cm. or the voltivity, and H is the drop in magnetic potential per cm. or the gilbertivity.

[1] So far as the ether is concerned any one of these theories is sufficient. It is only when we come to consider matter as well that we need a fourth equation.

This theory will probably be most familiar from Lodge's provisional use of it in his Modern Views of Electricity, as a means of giving a most lucid and satisfactory working conception of most of the phenomena of electricity and magnetism.

If we let $Z = M/L^2T$ we get the reciprocal theory to the above, *i. e.*, that in which $\varkappa$ is the density and μ the compliancy.

A very considerable number of theories has been treated of by Heavyside in his Electromagnetic Theory, Vol. I. That on page 254 may be obtained by letting $Z = M/L^2T$, and is thus the reciprocal theory referred to above, making $\varkappa$ the density and μ the compliancy, though this might not appear to be so at first glance, as that titanic mathematician is fond of using a special notation. The other difficulties which he refers to in connection with this theory, which in a modified form was first suggested by Kelvin, are no doubt those connected with the discontinuity of electric force at an interface.

Let $Z = ML^2T$, then,

$Q = ML^2$ = moment of inertia,
$P = 1/T$ = angular velocity.
$E = 1/T^2$ = angular acceleration.
$G = ML/T$ = angular momentum.
$\mu = 1/ML$ = reciprocal of moment of inertia.

We can get an idea of the mechanism of this theory by considering the motion of a pulley loosely fixed on a revolving shaft.

Let $Z = T^3/M$, then

$$Q = LT, \qquad P = ML/T^2,$$
$$\varkappa = T^4/ML, \qquad \mu = M/LT^2.$$

This theory makes the force with which an elastic string is pulled analogous to quantity of magnetism, the amount of stretch analogous to the gilbertance[1] and the elasticity of the string analogous to the magnetic permeability.

Reduction of the Number of Possible Theories.

So far we have not limited the values of Z in any way. It may contain any power, integral or fractional, of M, L and T.

[1] Gilbertance, difference of magnetic potential.

But from examination of equations 5–8 we see that if Z does not contain M, then Q and P cannot be quantities of the same nature as those met with in mechanics, but must be square roots of such quantities. But if we have two charged bodies in space, having equal charges, the bodies themselves consisting of elastic hollow conductors, there being an indefinitely small hole in one of them, if we let the other conductor shrink so that it can be introduced into the indefinitely small hole in the other, the one conductor will now have a double charge. If these conductors are by themselves in space, it is difficult to see why, if Q were the square root of some dynamic quantity, the charge should be twice the original charge.

The same reasoning applies to the supposition that Q or P may be some other power than the first of one of the quantities we meet with in mechanics, and, hence, we arrive at the result that Q and P either contain M to the first power or else M is absent. This makes Z to contain M to the power $+ 1$ or $- 1$, as may be seen from equation 5.

As regards the power of L, it is difficult, as Williams has pointed out, to understand how a meaning can be attached to powers of L higher than the third. L is a length, L^2 is an area, and L^3 is a volume, but L^4, in conjunction with M to the first power, is not at present understandable. Consequently we may for the present suppose that L does not enter into the qualities of Q, P, $\varkappa$ and μ to a power higher than the third. From examination of equations 5–8 we see that this means that L must enter into Z either as the $0 + 2$ or $- 2$ power.

But L cannot enter into Z as the 0 power, for since the capacity of a condenser has the quality $\varkappa L$ and the permeance of a magnetic circuit the quality μL, either capacity or permeance must have the quality $M^{\alpha}T^{\beta}L^{\circ}$, *i. e.*, must be entirely independent upon configuration. But we know that this is not true, for by changing the ratio of area to cross section we can change the capacitance or permeance and for a given change of ratio, the capacity is modified in the same way no matter what the substance or physical state. Consequently Z must contain L to the power $+ 2$ or $- 2$.

Again, from inspecting equations 5–8 we see that the exponent of M must have the opposite sign to the exponent for L, other-

wise either capacitance or permeance will have the quality ML^2/T^4 and either Q or P will have no dimensions, T having the index given below.

As regards the exponent of T, in order that the quality of Q or P may not contain a fractional power of T, the index of T must be odd. If we reject all powers of T above the fourth as improbable, then T must enter into Z with the index $+1$ or -1. For if it had the index $+3$ or -3, then either the time rate of change of current or of voltage would be the fifth differential with respect to time of some quantity. This may, of course, be true, but can hardly be considered probable.

As we do not, in mechanics, meet with any quantity in which the exponent of M is of the same sign as the index of T, except when they are both zero, we finally arrive at the result that the quantity of Z is

$$M^{\pm 1}L^{\pm 2}T^{\mp 1}$$

or, substituting in equations 7 and 8,

either κ is a density and μ a compliancy, or vice versa.

a result which is identical with that of Williams, though derived in a different manner.

It will be observed that in reaching this result we have not assumed *a priori*, that Q and P contain M to the 1 or 0 power, but have taken the additive property of Q and of P as our reason for considering such to be the case. To the writer this seems conclusive, but as will be seen later, the above proof is not at all essential to my reasoning, though it happens to agree with it, and I am myself persuaded of its validity.

PART II.

The Critical Relation.

We need a fourth equation. An account of some of the work done in past years, with a view to obtaining this equation, is given in an appendix. Instead, however, of examining phenomena, to see if they furnish the required equation, we may employ the converse method, *i. e.*, examine what equations are possible and see

if there are in existence phenomena which fit them. This is advisable for another reason. It is held by some, to whom I have explained my ideas, that phenomena cannot be predicted by means of qualitative mathematics. I might reply to this that the method has as a matter of fact, been successfully used for this purpose, but for the present I prefer to abandon that ground and to rest my argument on the fact that, whether phenomena can or cannot be predicted by means of qualities, yet when the phenomena are discovered, how or by what means it matters not, the qualities can be determined from them. In equations 1–3 we have relations between Q and P, and $\varkappa$ and μ, but not between Q or $\varkappa$ and P or μ. Now the relations between Q and $\varkappa$ and P and μ involve P and Q respectively, giving merely the relations between the forces, fluxes and resistances. Consequently, we are thus naturally led to look for relations between Q and μ or P and $\varkappa$.

As Q and μ belong to different systems, we must reduce them to the same. This we do by noting that Q/T is the current and that this is a measure of the gilbertance. But μ is a specific quantity, consequently we must take the gilbertance per unit length, or the gilbertivity, *i. e.*, $Q/LT = H$. Similarly $P/LT = F$ the voltivity.

We are, therefore, to look for a relation between H and μ or F and $\varkappa$.

We may write F, H, $\varkappa$ and μ in terms of M, L, T and Z, and by giving different values to Z, see what relations should exist if that value of Z were correct. This was done and many relations were examined, and the result was again reached, by methods which I am not prepared to defend, that, as either $\varkappa$ or μ must be a density. For the present, assuming that the previous work of Williams and the writer is correct and that Z is either M/L^2T or L^2T/M, we obtain as a result, either

$$F = L/T \qquad \varkappa = M/L^3$$
$$H = M/LT^2 \qquad \mu = LT^2/M$$

or

$$F = M/LT^2 \qquad \varkappa = LT^2/M$$
$$H = L/T \qquad \mu = M/L^3.$$

From this we arrive at the conclusion,

Of κ and μ, the one which is a compliancy will be an inverse function of the corresponding force.

This is our touchstone, this relation. We know at once that the capacity of a condensor does not decrease when we increase the voltage on it, whilst we know that the permeance of a magnetic circuit does decrease when we increase the gilbertance. It remains to be proven that the rate of this change is that called for by the equation.

The Variation of μ with H.

The existence of a relation between H and μ has long been noticed, and a number of attempts has been made to formulate it exactly. Until recently the most satisfactory formula was that proposed by Lamont and Frölich. This is

$$1/\mu = a + bH.$$

This however is obviously incorrect for high values of induction, as it makes the total induction reach a limit, and Ewing's experiments show that this is not the case. A satisfactory formula was however proposed by Kennelly.[1]

He divides the total inductivity B, into two parts. One part due to the ether in the space occupied by the iron, and numerically equal to H which we will call B_e. This,[2] since the value of μ for the ether may be taken as 1, can increase indefinitely as H increases. The other part of B, that due to the iron, which we will denote by B_i, he calls the ferric inductivity. The ratio between H and B_i he calls the ferric reluctivity, and we will denote it by ν_i.

Then, with these definitions, Kennelly's formula is

$$\nu_i = 1/(\mu - \mu_e) = a + bH \,(= \text{numerically } 1/(\mu - 1).)$$

That this formula is correct within the limits of experimental error has been shown by Kennelly himself for a number of magnetic materials.

Figures 1, 2 and 3 show the relation between ν_i and H for various samples of iron, nickel and cobalt. It will be seen that in every case the main part of the curve is a straight line, as it should be.

[1] Trans. Am. Inst. E. E., Oct. 27, 1891.

[2] I shall, to avoid considerable confusion and the use of a large number of additional symbols, take μ and κ as unity in vacuo.

This formula, it will be observed, does not hold for the first part of the curve. There is an abrupt bend at low values of H.

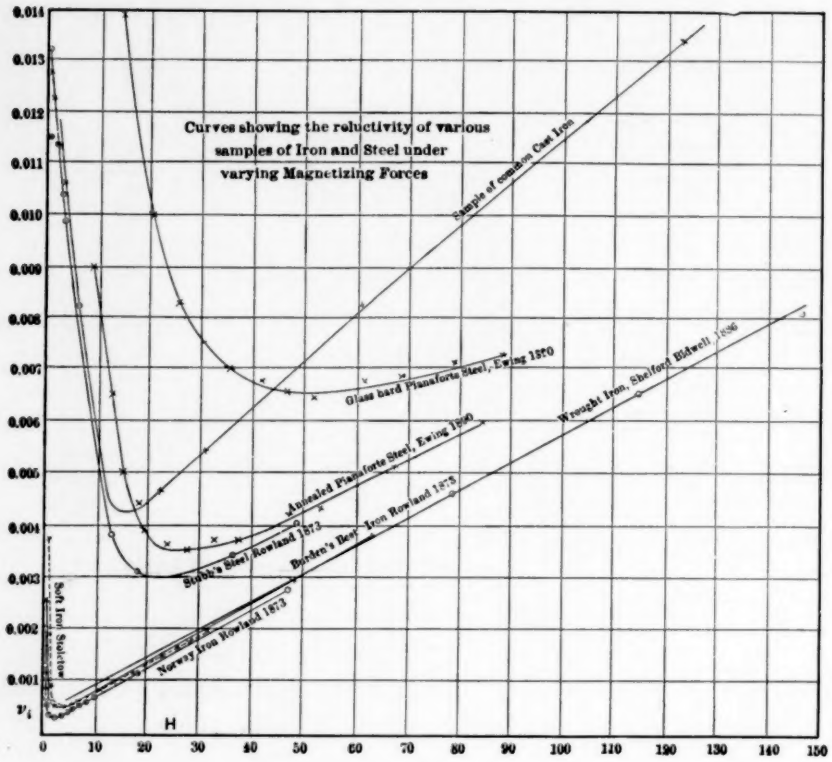


Fig. 1.

This bend however is not an essential part of the phenomenon; it disappears when the wire is tapped or has an alternating current passed through it. It seems to be due to the same cause as hysteresis, *i. e.*, molecular friction. It will be noted that those materials for which the hysteresis has a high value show the bend most pronouncedly.

Curve *A*, Fig. 5, shows the bend for a soft iron wire, and curve *B* is for the same specimen when an alternating current is passed through the wire. It will be seen that in the latter case the bend has entirely disappeared and the curve follows the formula throughout its length. The bend is therefore merely a superimposed phe-

nomenon. The experiments from which these curves were drawn are those of Gerosa and Funzi.[1]

The question may, however, be asked, and has been asked, "Is not this relation one peculiar to iron and the other strongly magnetic

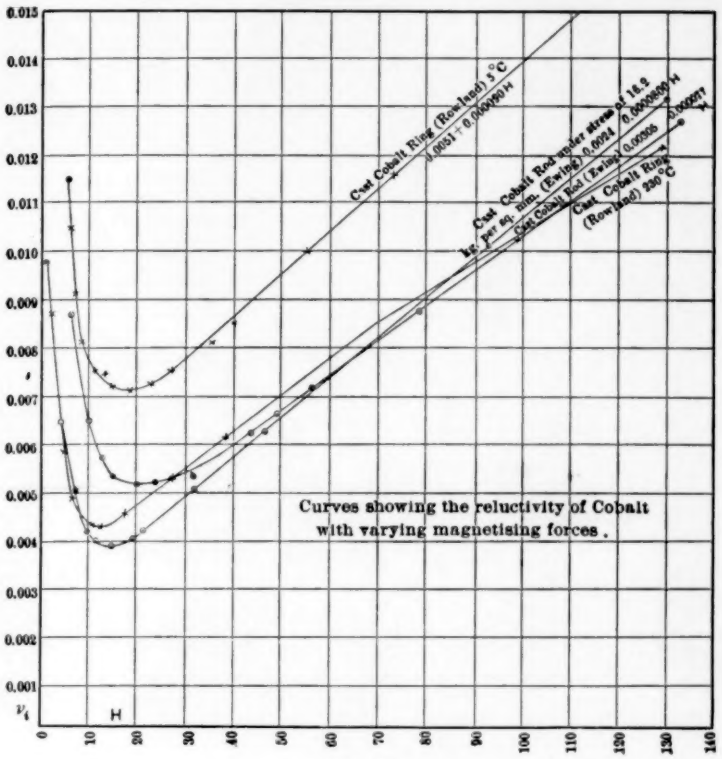

Cast Cobalt Ring (Rowland) 3 °C
Cast Cobalt Rod under stress of 16.3 kg. per sq. mm. (Ewing) 0.0024
Curves showing the reluctivity of Cobalt with varying magnetising forces.
H

Fig. 2.

metals?" This answer is, that it is not, but appears to hold for all materials. For Steinmetz has shown,[2] that the same law holds for alloys of iron, such as the non-magnetic 10 % amalgam of iron for which the permeability is as low as 2. As, however, these alloys all had a strongly magnetic component, the writer decided to investigate the magnetic relations of some of the very slightly magnetic materials. Fortunately, however, in view of the great dif-

[1] Recond. del. Inst. Lombardo. See Ewing's "Magnetism in Iron and Other Metals," p. 319.

[2] Trans. Am. Inst. E. E., Sept. 27, 1892.

ficulty of the task, it was discovered that Silow had published some tests made on a solution of chloride of iron[1] and on plotting his results it was found that this substance also showed the same phenomenon, *i. e.*, the reluctivity decreases at first, but as H increases

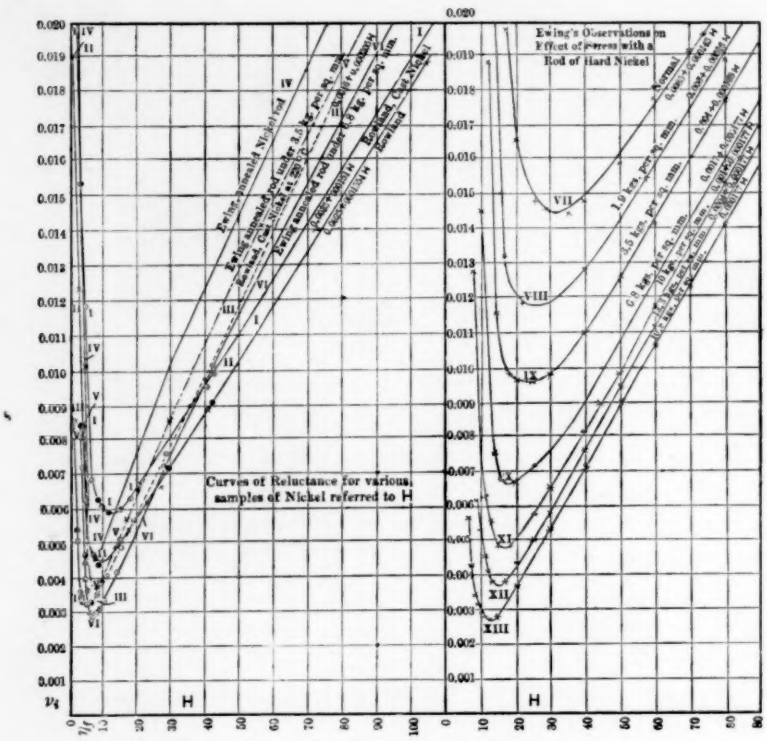


Fig. 3.

Fig. 4.

still further the reluctivity increases also. The subjoined table gives Silow's results, and it will be seen that they are of the same general nature as those obtained for iron though, possibly on account of the difficulties of the experiment, the curve is rather irregular. I propose to investigate this subject further.

Plucker also noticed the same thing in the case of bismuth. A set of measurements, ranging from $H = 1100$ and up by another experimenter did not show the effect, but the test should be made with values of H from 1 to 20, as even in iron the permeability is practically constant when H is large.

[1] Wiedemann, Annal., Vol. XI., 1880, p. 324.

Table Showing Variation of Susceptibility with H, for Ferric Chloride Solution.

H	Susceptibility.	H	Susceptibility.	H	Susceptibility.	H	Susceptibility.
1.15	0.000096	1.81	0.000142	2.40	0.000099	6.54	0.000065
1.35	0.000104	1.90	141	2.45	104	7.00	062
1.60	131	1.96	131	3.73	070	10.00	060
1.70	131	2.13	111	5.33	069	12.60	055

As the relation is shown to hold for substances for which ν_i has such different values as 0.003 (iron), 0.004 (hard steel), 0.1 (iron filings)[1] 0.9 (11 % iron amalgam) and 1000 (ferric chloride

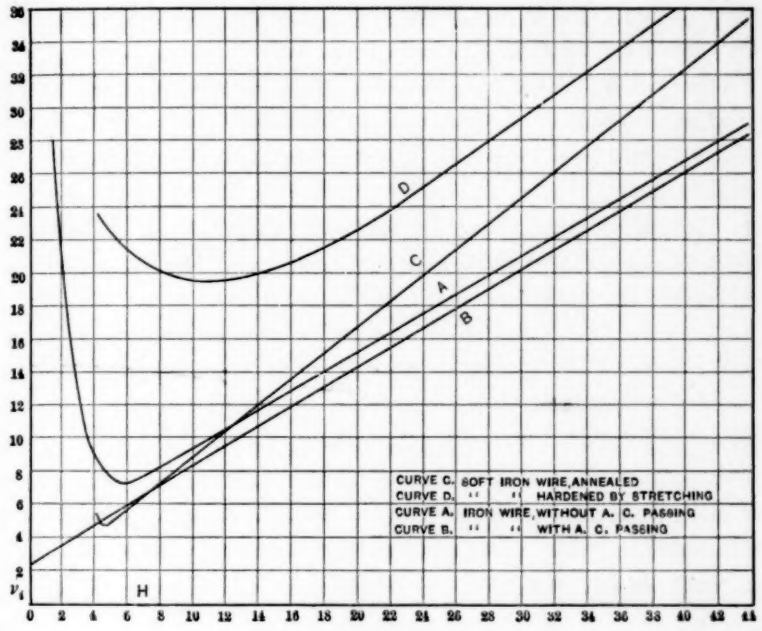


Fig. 5.

solution), it is extremely probable that the law holds generally, and that we are justified in making the following statement:

The relation between H and ν_i is given for all substances by the formula

$$\nu_i = a + bH$$

or, in a different form.

[1] See Steinmetz, ibid.

For all substances,

$$dB_i/dH = aB_i^2/H_i$$

where, as will be seen later, a is a constant connected with the elastic properties of the material.

Having shown that the initial bend of the curve can be removed, and is thus merely a superimposed phenomenon, and that the law holds, in all probability, for all classes of substances, the question next arises, "Is the relation on exact one?"

Unfortunately we have at present no record of any very exact tests on iron or similar materials. That is to say, not exact as regards the connection between B and H, though no doubt amply precise enough for the purpose for which the tests were made. This is for the following reasons:

The experiments have been made on either straight wires, rings of iron wire, solid rings, solid rods or laminated rings. In all cases there are sources of error which were not taken into account in the experiments.

In the case of straight wires, the permeability varies with the induction, consequently the leakage varies also. This may give rise to errors of as much as 20% for high magnetizations.

In the case of rings of iron wire, no accurate results can be obtained, as the flux, having to pass from turn to turn through air, varies very greatly in density and the reluctance of the air spaces cannot be calculated owing to the varying permeability of the iron. Steinmetz has shown[1] that the variation from the mean induction in iron wire, wound somewhat differently, may be so great as to increase the hysteresis 900%. Such tests are not in any way a measure of the quantities supposed to be involved.

In the case of solid iron rods the induction cannot die away quickly owing to the eddy currents set up. Another still more serious error, especially with very thick rings, is the fact that the length of the magnetic circuit on the inside of the ring is less than that of the outside of the ring. If the ring is one foot in diameter, and the ring one inch thick, then this difference will amount to 16%. In other words the value of H in the inside of the ring is 16%

[1] Trans. Am. Inst. E. E., pp. 699 and 701.

greater than that in the outside. As the permeability is not constant, the mean permeability is not the same as the permeability of the mean circumference, and this difference may amount to several per cent. Though this source of error was pointed out by Kennelly[1]

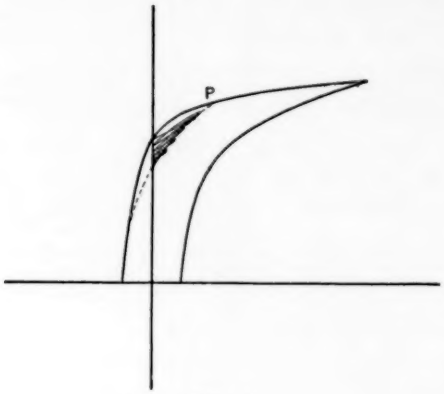


Fig. 6.

I know of no published experiments in which this very important source of error has been taken into account.

With solid iron bars tested in a permeameter there are quite a number of sources of error. But there is one thing alone which would vitiate all results obtained, especially for high values of B for which the permeameter is commonly supposed to give its most accurate readings, and that is that the formula used is incorrect.

The pull per sq. cm. is commonly given as

$$\text{Force} = (B^2 - H^2)8\pi.$$

It should be

$$B^2/8\pi\mu_m$$

where μ_m is the permeability of the medium, generally air, which flows in when the gap is made. The proof of this formula I have given elsewhere.[2] It is readily seen that the usual formula is wrong, for if we perform the following cycle :

1. Pull the bar, leaving the magnetizing coil in place.

2. Fasten the coil to the bar and let the bar come back to its original place.

[1] Electrodynamic Machinery, p. 28.

[2] Frank. Inst. Lecture, Magnetic Design, April, 1899.

3. Move the coil to its original place.

It is self evident that if the pull be given by $(B^2 - H^2)/8\pi$ we will have a perpetual motion machine.

It is for the above reasons that the writer was reluctantly forced to the conclusion that there was no published work which could be relied upon to settle the point whether iron obeyed Kennelly's formula with an accuracy of one per cent. or less. A very elaborate and careful test was then made, the results of which are shown in Fig. 7, the method being described in an appendix, and the conclusion reached was that:

The sample of soft iron tested obeys the formula, with a maximum deviation from it of less than ¼%, this being the maximum amount of experimental error, for values above the bend of the curve.

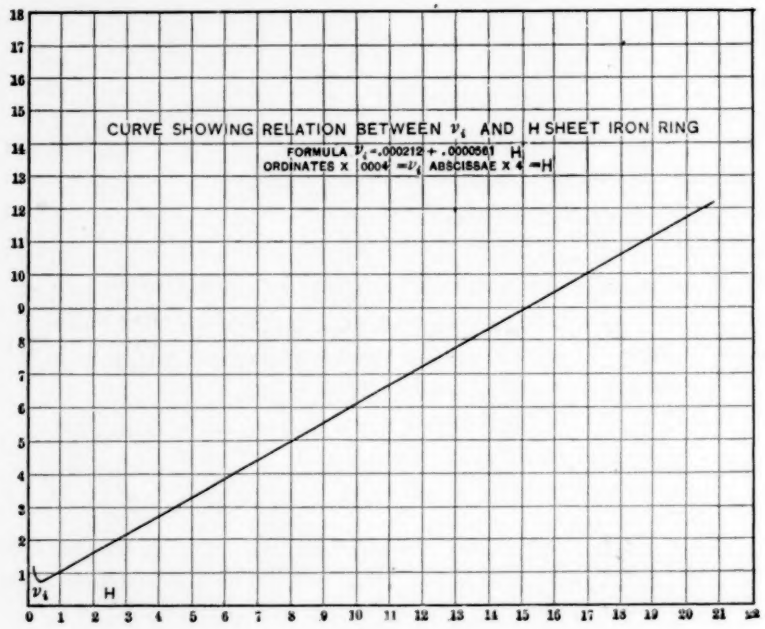

CURVE SHOWING RELATION BETWEEN ν_4 AND H SHEET IRON RING
FORMULA $\nu_4 = .000212 + .0000561\ H_4$
ORDINATES X .0004 $= \nu_4$ ABSCISSAE X 4 $= H$
ν_4
H

Fig. 7.

The Solution.

To resume, we have seen,

1. *That if the electrical quantities are of the same nature as those met with in mechanics, either κ must be a density and μ a compliancy, or vica versa.*

2. *That, this being the case, either* $\varkappa$ *must decrease as* F *increases, or* μ *decrease as* H *increases.*

3. *That whichever one of the two quantities,* $\varkappa$ *or* μ*, decreases as the corresponding intensity increases, that one must be a compliancy.*

4. *That* $\varkappa$ *does not decrease as* F *increases, but that* μ *does decrease as* H *increases.*

5. *That the rate of decrease is in agreement with the qualitative formula.*

6. *That therefore* μ *is of the nature of a compliancy.*

Z must therefore have the value M/L^2T. On substituting this value in equations 5–8 we obtain,

$$(12.) \qquad Q = M/T$$

$$(13.) \qquad P = L^2$$

$$(14.) \qquad \varkappa = M/L^3$$

$$(15.) \qquad \mu = LT^2/M$$

Part III.

Corroborative Evidence. A. Permeability.

Since, as we have seen

$$\nu_i = a + bH$$

and ν_i, being the reciprocal of a permeability, has the dimensions M/LT^2 then a must have the same quality.

Therefore a should vary with the elasticity of the material.[1] We can see that this is true by examining the curves of Fig. 1. Here the very hard materials, like pianoforte steel have large values of a. The soft materials, such as soft wrought iron have very low values of a. Intermediate materials, like cast iron have intermediate values of a.

Again if the material be hardened by stretching, a is increased. This is shown from the curves in Fig 5, taken from the experiments of Ewing.[2]

[1] This may be questioned. I will not defend my statement here, as the main matter of importance is the fact that it *does* vary, whether the prediction of the phenomenon was or was not legitimate.

[2] Ibid., p. 304.

It will be noted that though a has changed markedly, b remains unchanged.

Again, M/LT^2 is of the same quality as a stress. Consequently we should expect a to change when a stress is applied to a wire. The curves shown in Figs. 4 and 2 show that this is true for both nickel and cobalt. These are drawn from Ewing's experiments,[1] and are taken from Kennelly's paper.[2] That the same thing holds true for iron may be seen from Ewing's experiments on that substance[3] though the experimental data are not very numerous.

In some of the curves given, it would appear as if b also changed slightly. This may be true, but it may also be due to the fact that when the bend is very gradual it is difficult to calculate the value of b from the data given, as the line remains inflected for some distance past the bend. The direction of the change of a should be noted as it gives us an idea of the nature of the stress.

M/LT^2 has the same quality as η in Steinmetz's formula for hysteresis, for both have the quality of work per unit volume. We might therefore expect η to change with a. The accompanying table[4] shows that this is true. Note especially the cases where the same material is tested in the annealed and hardened states. In each case a and η change, whilst b remains constant.

b has zero dimensions, and I have found that there is a relation between a, b and η as follows,

$$\frac{b\eta}{a} = 0.007, \text{ approximately.}$$

There is some evidence in favor of the idea that b has but two values for iron, one for soft iron and one for cast iron, with intermediate values when the material is intermediate in quality. It may therefore be related to the chemical constitution of the sample, and may represent the reciprocal of the number of molecular groups. There is not sufficient data at present available to enable us to decide the question.

[1] Ibid.
[2] Ibid.
[3] Ibid., p. 179.
[4] Taken from Steinmetz's paper, Ibid.

Table Showing Relation between a, b and η for Various Substances.

Material.	a	β	η	$(\beta\eta/a)\times 10^{6}$
Wrought Iron, Norway	.166	.054	0.00228	742
" " Common	.20	.055	0.00326	896
Sheet Iron	.275	.058	0.0035	738
" " Ferrotype, Commercial	.45	.050	0.00548	609
" " " Annealed	.337	.050	0.00458	679
" " Tinplate	.192	.055	0.00286	819
" " "	.321	.053	0.00426	703
Very Soft Iron Wire, Erwing	.20	.064	0.0020	640
Cast Iron No. 1	2.40	.094	0.0130	509
" " " 2	2.43	.094	0.0132	510
" " " 3	2.76	.095	0.0158	544
" " " 4	2.05	.097	0.0113	535
" " " 5	2.34	.095	0.0127	516
" " " 6	2.07	.097	0.0122	572
" " " 7	2.37	.098	0.0136	562
" " " 8	2.92	.095	0.0146	475
Cast Steel, Hardened	2.7	.054	0.0280	560
" " Annealed	.88	.054	0.00848	520
" " "	2.00	.091	0.012	546
" " Average of 5 samples	.736	.057	0.009	679
" " " "	.35	.054	0.005	771
Mitis metal	.30	.054	0.00428	770
Welded Steel, Tool Steel HI	8.0	.121	0.0748	1133
" " " " H	7.8	.105	0.0613	825
" " " " OH	1.9	.066	0.0267	927
" " " " O	1.54	.060	0.0270	1055
" " " " Sh	1.33	.060	0.0190	857
" " " " S	1.22	.058	0.0145	689
" " " " Nickel	1.00	.170	0.0122	207

Corroborative Evidence. B. Specific Inductive Capacity.

Since μ is a compliancy, $\varkappa$ must be a density and its quality must be M/L^3

The fact that dense substances are generally highly refracting was first pointed out by Newton. Diamond and Faraday's dense glass are examples, as are also those liquids used for making determinations of the densities of heavy minerals. The following table shows the relation plainly. The data are taken from Everett's C.G.S. Units, and Landolt and Bornstern's tables. The tests on the glasses were made by Hopkinson; those on the gases by Ayrton and Perry.

Substance.	Density.	$\kappa - 1$.	$(\kappa - 1)$ / Dens.
Glass.	4.5	9.1	2.0
Glass.	3.66	6.4	1.75
Diamond.	3.5	5.1	1.46
Glass.	3.2	5.85	1.82
Glass.	2.87	5.57	1.59
Sulphur.	2.	2.84	1.42
Benzol.	1.29	1.58	1.22
Rubber.	.92	1.32	1.43
CS_2.	.88	1.36	1.54
Paraffin.	.87	1.29	1.48
Kerosene.	.8	1.07	1.34
Petroleum Spirit.	.7	.92	1.31
SO_2.	.00269	.0052	1.93
CO_2.	.00195	.0023	1.18
Coal gas.	.00125	.0019	1.52
Air.	.00127	.0015	1.18

It will be seen that in the case of these substances, though the densities vary from 4.5 to 0.00125, or in the ratio of 4000 to 1, the added κ is roughly proportional to 1.5 times the density.[1]

As this table might give a wrong impression, it should be stated that though the bodies taken were chosen at random, yet several substances, such as water and the alcohols, which behave anomalously in other ways, have been omitted. Some of the discrepancies may be accounted for. Thus I have found that the capacity of most of the organic oils is considerably decreased when they are carefully purified from organic acids, mucins and water. In the case of hydrogen the discrepancy may be due to the difficulty of measuring such a very small capacity.

But we cannot expect this relation to be more than an approximation for the following reason: The value of κ varies with the periodicity and hence it is certain that, as in the case of the stretched string, the virtual mass must be made up of several terms. In fact, J. J. Thomson has shown that the capacity of a moving sphere is increased in the ratio $1/(1-4/15\,\mu\kappa V)$. So that in addition to the density, the expression for κ for a substance must contain terms depending upon the valency, the number of atoms per cc. the velocity of the atoms and the periodicity of the electric stress. These other

[1] Newton gives a similar table (Opticks, Prop. X.).

terms may, it is conceivable in certain cases, be of larger numerical value than the original density term.

Viewed, however, not as an attempt to find a law for refraction, but merely as a broad test between two rival theories of electricity, it is evident that $\varkappa$ is related to the density and not to the elasticity of the substances.

As regards the variation of $\varkappa$ in any particular substance with change of state, we have Newton's, Gladstone's, Lorenz and Lorentz's and Ketteler's formulæ. The first three of these do not agree satisfactorily with experiment. The last one does, probably because it contains more constants.

The series in Ketteler's formula can be expressed as a fractional power, and hence is of unity dimensions. This makes the refractive index to vary as the density, and since μ is practically unity for all the substances to which this formula applies, we get $\varkappa =$ density. Not much importance can be attributed to this result however.[1]

It is to be noted that the refractive index of a gas seems to vary in the direct ratio as its density. But the denser the gas the closer the atoms must be to each other, and the greater the molecular attraction forces, according to Van der Waal's law (or the writer's modification of it, *i. e.*,

$$\left(p + \frac{c}{v^{\frac{4}{3}}}\right)(v - b) = R\theta\,).$$

Hence, also, in all probability, the greater the permeability. But, unfortunately, we know too little about these molecular actions at present to enable us to use the phenomenon in this connection.

Determination of the Density and Elasticity of the Ether.

If the ether has specific mass, and from the phenomena of light it must have some equivalent property, how would it affect the refraction of light?

In the first place we know that for most substances μ, the compliancy is practically the same as for the free ether. This means either that the elasticity of the free ether must be the same as that

[1] These considerations suggest the following formula: $\frac{n^2 + 1}{\text{density}} + A = \text{Constant}$, where A represents terms depending on the velocity, valency of the atoms, etc.

of the substance, or else that the total area of cross-section of the atoms is very small compared with the cross-section of the space they occupy to the exclusion of other atoms, *i. e.*, that the atoms have configuration virtually equivalent to that small framework structures of thin platinum wire interpenetrated by another substance.

If we are to look upon the atoms as possessing a structure whose actual cross-section is small compared with that of the space occupied by them,[1] their effect on the elasticity and density constants would be analogous to that obtained by adding to a vessel filled with oil a number of small structures built out of platinum wire. In this case the cross-section of the oil not being appreciably changed, the elasticity would not be altered to any extent, whilst the density of the platinum wire structures would be added to that of the oil. The resultant velocity of wave propagation in the space occupied by the combination would therefore be given by the formula

$$v = R/(d + D)$$

where d is the density of the oil, D the mass of the platinum wire structures contained in one cc. of space, and R the elasticity of the oil.

In this case, if the oil were set in motion with a velocity v, the velocity of the waves through it would be increased by the amount

$$vD/(D + d).$$

This fraction is of the same form as that called for by the aberration of light and Fizeau's experiment. Consequently if the previous work be correct and $\varkappa$ is a density, then the fraction $(\varkappa - \varkappa_0)/$ density,[2] should be constant for all substances, at least to a first approximation.

Now the above table shows that this is approximately the case. There are exceptions, such as water and the alcohols, but by far the

[1] The preceding table was prepared before I noticed the great discrepancy between Ayrton and Perry's and Boltzmann and Klemencic's values of κ for gases. If the values given by the latter be correct, the atoms may be comparatively solid, and we obtain 1.33 as the density and 12.10^{20} as the elasticity of the ether. As this is not yet absolutely certain, I leave the above as written.

[2] Taking κ for the ether as unity, this is numerically equal to $(\kappa - 1)/$ density.

greater majority of substances obey this law. We know so little about the constitution of the atom that we cannot at present discuss the peculiar behavior of the exceptions. It is to be noticed that those bodies which do not agree with Maxwell's [1] law, that the refractive index for light is $\varkappa^{\frac{1}{2}}/\mu^{\frac{1}{2}}$, are those which also disagree most widely from this rule that $(\varkappa - \varkappa_e)$/ density is a constant.

The determination of an approximate value of the density of the ether from this formula is a very simple matter. Since in the formula, $\varkappa$ is the specific inductive capacity of, or density of, both the stuff and of the space occupied by it, and we take the specific inductive capacity of the ether, according to our present system of units, as unity, then $\varkappa - 1$ is the density of the material in terms of the density of the ether. The average value of the constant is 1.5, therefore the value of the density of the material in terms of the density of the ether is 1.5 times its density in terms of water, *i. e.*, the density of the ether is 0.66, or two-thirds that of water.

From this, by means of the formula for the velocity of light, we may deduce the elasticity of the ether and find it to be

$$R = 6 \times 10^{20}.$$

As a converse proposition, it follows that to find the specific inductive capacity of any substance, all we have to do is to add to its specific mass the specific mass of the ether, and then divide the sum by 0.66. Thus, to get the specific inductive capacity of a mineral oil of density 0.5, we get

$$(0.5 + 0.66) \div 0.66 = 1.75 = \varkappa.^{2}$$

It will be noted that the values given for the constants of the ether lie midway between those generally given. Lord Kelvin's value for the elasticity[3] is 900, with a correspondingly small density.

[1] I have previously pointed out that those substances which do not obey Maxwell's law act as negative uniaxial crystals in Kerr's experiment, whilst those obeying the law act as positive crystals, and that this indicates that those acting as negative crystals probably contain a certain per cent. of particles of substances of large specific induction capacity. —Elect. World, Jan. 2, 1897.

[2] Accepting Boltzmann's and Klemencic's results for κ we have, Mass of atoms in one c.c. — density of ether (1 — volume actually occupied by atoms) $= \kappa$. Hence, for spherical atoms not in the form of crystalline solids, we have the formula $(n^2 - 0.7)$/density = constant.

[3] Trans. R. S. E., XXI., 60.

This value, however, can hardly be correct. For unless the atoms hold together by means of hooks of some kind, it would seem as if the stress when a bar of metal is sheared must ultimately come on the ether. In fact, the writer has shown[1] that the phenomena of rigidity, elasticity and tensile strength are quite well explained by supposing that the atoms are electric doublets and have charges on them of the same size as those called for by Faraday's law of electrolysis, and that the effects observed are of the same dimensions as those called for by this theory. But whether this theory be correct or not, it would seem that if the ether is the thing ultimately acted upon, it must be at least as rigid as the most rigid substance we know of, *i. e.*, the diamond, which has a rigidity of approximately 10^{14}. This calls for a minimum density of the ether of about 10^{-7}, or approximately that of air at 1/15 mm. pressure. Therefore, on the very lowest supposition the density of the ether in a vacuum tube must be immensely greater than that of the enclosed gas.

The other hypothesis, that held by Fitzgerald and by some other eminent naturalists, makes the elasticity of the ether to be about 10^{21} and its density at least 20. At first sight it would appear extremely reasonable to suppose that, if the atom be simply some configuration of the ether, the ether must be at least as dense as the densest atom. More especially since, as has been pointed out, both the fact that the permeability, *i. e.*, elasticity of the ether, is but very little affected by the presence of matter and the fact that, (since the fraction $(\varkappa - \varkappa_0)/$ density is approximately constant, *i. e.*) the density of the element is simply added to that of the ether, seem to show that the virtual cross section of the atoms is a small fraction (say one per cent. or less) of the space occupied by the atoms to the exclusion of other atoms. (Of course the actual space may be greater but in this case there must be some compensating device brought into play.)

But, on the other hand, it is not evident that there is any *necessary* connection between the density of the ether and the density of the atom. For it is very easy to see that if the atom consisted of some

[1] Elec. World, August 8 and 22, 1891, more fully, Science, July 22, 1892, and March 3, 1893.

kind of a vortex in the ether, that owing to some property of the atom, say its energy of rotation, a strain might be set up in the ether proportional to its velocity squared. The total density of the atom might, therefore, be made up of two parts, one due to the density of the ether itself, which might be extremely small, and the other depending in some way on the configuration of the atom, this latter part being associated with a proportional quantity of gravity. Such an hypothesis, however small its justification, enables us to see that the ether *need* not be as dense as platinum.

Again, we may take a vortex ether, such as has been treated of so powerfully by Fitzgerald and Kelvin. In this case we may conceive of space as filled with a tangle of vortices, the actual volume of the vortices being small compared with the space they occupy, this tangle of vortices being what we call the ether. If, by any action a new configuration is made to take place, and this is done in such a manner that it has the properties of matter, it may, at the same time, call for a crowding together of the vortices at that portion of space. In fact there are a number of, not at present improbable, suppositions which we might make for the purpose of showing that there is no *necessary* connection between the density of the ether and of the elements.

(*To be continued.*)

POLARIZATION AND INTERNAL RESISTANCE OF THE COPPER VOLTAMETER.

BY B. E. MOORE.

MEASUREMENTS of electrolytic polarization may be divided into two general classes. In the first class may be placed all the measurements made with the direct current. This class includes the greater part of the determinations. In the second class the alternating current has been employed. The development of this method is a comparatively recent one, but promises to yield extremely important results. This method has also afforded measurements of electrode capacity, designated by "polarization capacity"—an expression which first appeared in a contribution by Kohlrausch.[1] This term suggests that the phenomenon is limited to the electrodes and to the solution immediately adjoining them. The enormous values obtained for the capacities,[2] indicate strata so thin as to be scarcely designated as finite. Their values are a reminder of the Helmholtz charged strata theory. However, the Warburg theory, which makes it a current conduction, and at the same time recognizes that both the electrodes and the solution play important parts, is more satisfactory. The process at the electrodes is frequently complicated. Gases may be deposited and cover parts of the electrodes and thus insulate parts of the surfaces, or the gases may be occluded and form an opposing gas battery. There may be an absorption of ions in the surface of the electrodes. The liberated products may form at the surface of the electrodes a new material or the same material in a different state of aggregation.

[1] F. Kohlrausch, Pogg. Ann., 148, p. 443, 1872.

[2] M. Wien, Wied. Ann., 58, p. 37, 1896.
C. M. Gordon, Wied. Ann., 61, p. 1, 1897.
A. M. Scott, Wied Ann., 67, p. 388, 1899.
E. Warburg, Wied. Ann., 67, p. 493, 1899.
E. Newmann, Wied. Ann., 67, p. 500, 1899.

The solution may act upon the electrodes either upon open or closed circuit—an action which may be normal or arise by reason of impurities, either in the electrodes or the solution. The character of the solution at the electrodes when carrying a current may be changed sufficiently to cause action. A concentration series always arises owing to the unequal rates of migration of the ions. Any of these changes give rise to a polarization. These deposits upon or in the electrodes are frequently small and a slight variation in their total quantity affects appreciably the electromotive force, when these deposits are present at the electrodes, before the current is started. Their ions are also present in the adjacent solution, and the current affects their magnitude; therefore, the concentration series in the solution is always affected. After removal of the current the electrodes and adjacent strata of the solution would tend to return to the former state of equilibrium, and the concentration series would be removed by diffusion. However the process of diffusion is a rather slow one and could only appear rapid when the concentration strata are minutely thin. If the current had suddenly been reversed, it must have aided the return to the original condition and quickly established the same phenomena as before, but at the opposite electrodes. This is what probably happens when alternating currents are used under proper conditions. This makes the phenomena reversible, and upon its reversibility depends the accuracy of the electrolytic resistance, by means of alternating currents.

The observations for electrolytic resistance by means of the direct current, unlike the observations by means of the alternating current, have never given uniform values. The resistance is always larger when the current is small. The method of measuring electrolytic resistance by direct current involves the measurement of three quantities, viz. electromotive force, E, at terminals of the electrolyte; the current, i; and the polarization, P. The electrolytic resistance, r, is then calculated by the formula $r = (E - P)/i$. The determination of E and i is easy and accurate. However, P is not usually measured with the direct current still flowing. The circuit is broken and the voltameter thrown upon an independent current. The process requires time and a residual part only of

the polarization as measured. Wiedeburg has given an experimental formula[1]

$$P = \frac{P_1}{2}\left(1 - E^{-\frac{B_1}{S_1}q}\right) + \frac{P_2}{2}\left(1 - E^{-\frac{B_2}{S_2}q}\right)$$

where P is the total polarization. P and P' the maximum polarization at electrodes 1 and 2 respectively. s denotes surface of electrodes, B_1 and B_2 are constants depending upon the material of the electrodes, and q is the quantity of electricity which has passed.[2] Guthe has subjected the formula to an accurate experimental test. His experimental conditions made $P_1 = P_2 = P/2$, $B_1 = B_2$, $S_1 = S_2$ and the value of P so obtained, substituted in the simple formula above, gives the uniform values for electrolytic resistance. However, the actual measurement of P or an approximation to such a measurement is very desirable. As has been pointed out by several observers, the nearer the time from the opening the one circuit to the closing of the other approaches zero, the more accurate the values obtained for P, and therefore the more accurate the calculated value for r.

The object of the following experiments, is to emphasize these conditions and to show the growth of polarization upon closed and open circuit. It has therefore been necessary to employ the direct current. P has been measured between .00006 second, and .045 second after the polarizing electromotive force was removed. The latter has also been given wide ranges both in its value and in the time during which it was applied. The smallest time intervals, so far as could be learned, are ten times shorter than any previous experimenter has used.

Fig. 1 gives a diagram of the connections. The circuits were all made and broken by a swinging pendulum. The minute lateral motion of the pendulum was a troublesome factor until the steel cone pivots were drawn so tight that the pendulum ceased vibration after a few swings. The keys 1, 3 and 4 had micrometer screw attachments so that they could be adjusted accurately in the same plane as key 2. To obtain the deflection with the standard cell, B,

[1] Wiedeburg, Wied. Ann., 51, p. 2, 302.
[2] Guthe, PHYS. REVIEW, Vol. VII., p. 193.

key 1 is fastened open with a strip of glass, keys 2 and 4 are set, the six-point switch is turned to the left and the key *b* is closed by hand about the instant the pendulum is released. This was necessary since the time period from key 3 to key 4 or from key 1 to key 4 was not long enough to allow the condenser to assume the potential of the standard, owing to the large internal resistance of this cell. To obtain the total deflection from the storage cell the

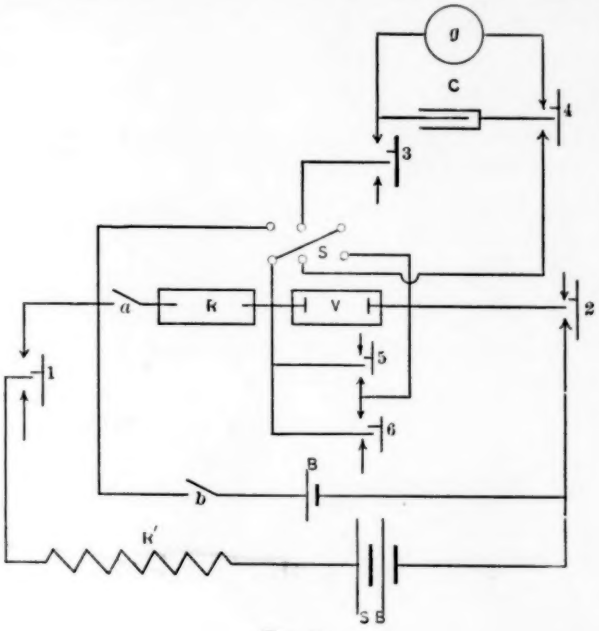


Fig. 1.

manipulation is the same except key 1 is used instead of key *b*. To polarize the voltameter key *a* is closed and keys 1, 2, 3, 4, 5 and 6 are set. When the pendulum is liberated, it first meets key 5 and opens the voltameter short-circuit. The detent to key 6 is overturned last of all and this again short-circuits the voltameter. By this swing of the pendulum the galvanometer will give a deflection proportional to the fall of the potential over the resistance *R* or the voltameter *V*, according as the six-point switch is turned to the left or right respectively. This fall of the potential over *V* will be the polarizing electromotive force, if keys 3 and 4 are released before key 2. It will be the polarization electromotive force if keys 3 and 4 are

released after key 2. The deflections over R and V are also proportional to the resistance R and the apparent resistance R_v of the voltameter respectively. The deflection caused by the polarization of V, subtracted from the deflection produced by the applied electromotive force leaves a value proportional to the internal resistance r. Whenever deflections are thus compared it is necessary that the time intervals of charging be the same. For the apparent capacity of condensers vary when the time interval is very short.[1] The capacities of a one-third microfarad condenser and a one microfarad condenser were found to vary about four per cent. while a Stanley condenser varied three fold. Key 3 was placed so that it made the up contact after the detent to key 2 was released. The equation of motion of the pendulum and the distance key 4 was displaced from the plane of key 2, determines the time which has elapsed since the charging circuit was broken. To release keys 3 and 4 before key 2, a small hard rubber block was attached to the pendulum opposite keys 3 and 4. The current was determined from the total electromotive force of the storage battery and the total resistance of the circuit. When the currents were large it became necessary to measure the electromotive force of the storage cell upon closed circuit, since a concentration series[2] arises in that cell. A comparison of the deflections of the standard and storage cell gives the electromotive force of the storage cell. A deflection is then obtained from the storage when the time interval is short. Let this deflection be D'. Then, if d_r and d_v be the deflection on closed circuit over R and V when the condenser is charged the same length of time we have

$$(1) \qquad D' = \frac{(d_r + d_v)(R + R_r + R')}{R + R_v}$$

where D' is proportional to electromotive force of the storage cell, and R' the resistance in series with R and R_v. The internal resistance of the storage cell is neglected. Farther (2) $R_v = R\frac{d_v}{d_r}$. This value of D' differs slightly from D for large currents. However, this correction was only made in the first experiment, and for the others the electromotive force of the cell on open circuit was used.

[1] PHYSICAL REVIEW, Vol. IV., p. 238.
[2] PHYSICAL REVIEW, Vol. IV., p. 353.

The adjustment of keys 2, 3 and 4 for a very short time interval was an extremely tedious task. Keys 2 and 4 were down contacts and the best method to adjust them was found to be by the sense of touch. The detents were held with the fingers against the spring contacts and the pendulum drawn back through an arc of about 2 to 5 mm., and allowed to fall against them. The time sense in the touch of the fingers is so delicate, that, by this method, a distance of .007 mm. could be detected. When the pendulum swung through its full arc the time required to pass through this distance at the lower point of the arc was .00002 second. The smallest time between keys 2 and 4 actually used was .00006 second, and in this time interval there is liable to be an error as large as twenty per cent., in a few cases. This error arises from the inaccuracy of the adjustment, temperature changes in the keys, lateral vibration in the pendulum, and any displacement of the keys due to pressure in making the settings. This setting could not be made one day and be found in adjustment the day following. Neither could it be made and key 3 adjusted to fall with its up contact between keys 2 and 4. Key 3 with an up contact had to be adjusted electrically and after much use it was found that its position would depend upon the electromotive force employed. The adjustment of keys 2, 3 and 4 for such a time interval requires three hours' work and then the adjustment could not be expected to remain over one half hour. After long trial, this short time interval was abandoned. A trial made it about .0005 second, revealed the fact that key 3 could be dispensed with so far as the polarization phenomena were concerned. The readings were the same, after allowing for difference in time interval of charging the condenser through the external resistance R, when key 3 was set as when it was allowed to remain against its up contact. In the latter case the condenser becomes charged to the polarizing electromotive force, and then upon open circuit discharges through the voltameter until it reaches the polarization electromotive force. This discharge would tend to maintain the polarization if the quantity is an appreciable part of the "polarization capacity" of the voltameter. Farther the electromotive force of the condenser would not have fallen to the polarization electromotive force if the time interval was sufficiently short. This time

interval must be determined from the equation of discharge of a condenser $q = \left(Qe^{-\frac{t}{RC}}\right)$ or $d = \left(De^{-\frac{t}{RC}}\right)$ where D is the total deflection of a galvanometer produced by a given electromotive force. E applied to a condenser of capacity C, and d the deflection of the same galvanometer after the condenser charged to the same electromotive force has been allowed to leak through a resistance R for the time interval t. A test of this kind was applied by charging the condenser over the resistance R made approximately equal to the resistance of the voltameter, and discharging it after leaking for an interval t. It was possible in this way to adjust the keys 3 and 4 so as to obtain an appreciable deflection, d. It was, however, not possible to obtain a deflection when the time interval was as long as .00006 second, provided the electromotive force of polarization was small. When the electromotive was large the spring in .00006 second is not sufficiently far removed from the lower contact to break the primary circuit. This process therefore became an excellent independent method to use as a check upon the settings of the keys. When the condenser is charged by the polarization electromotive force, after the polarizing electromotive force is removed, it lowers the potential of the polarization electromotive force, if the charge in the condenser is an appreciable part of the total charge in the voltameter, according to the law of divided capacities. For the time periods longer than .0006 second, a difference of two per cent. could be accounted for by experimental errors, but if there had been a difference of four per cent. in the readings, it could have been stated with some assurance that the capacity of the voltameter was between twenty-five and one hundred times as great as the capacity of the condenser. No such difference was detected and key 3 was allowed to remain against the up contact for all the recorded measurements and was only set when it was desired to make a test for capacity. No such effect was ever found with certainty but it seems reasonable to anticipate capacity measurements when the surface of the electrodes is reduced considerable in magnitude. After omitting the setting of key 3 the adjustment of keys 2 and 4 was greatly facilitated, and it became again possible to set to .00006 second, when the electromotive force was small. Wires not shown in the figure also lead from the volta-

meter to an alternating current bridge where the electrolytic resistance was determined by the Kohlrausch method.

The density of the solution used was 1.1395 throughout the experiments. The evaporation from the surface was compensated by adding distilled water. The temperature varied about four degrees in the first experiment, but less in the others. All solutions, except the first, contained about one-tenth per cent. of alcohol. The object of the alcohol was to prevent oxidation of the electrodes, but in this proportion it was not a success. The solution was also boiled to free it of the contained oxygen. Fresh solution from a stock bottle was taken for each experiment. The results of the first experiment are contained in Tables I. and II., and Figures 2, 3 and 4. The copper plates had stood in the solution for an entire

TABLE I.

$i.$	$E.$	R_V	$i.$	$E.$	R_V
.00188	.106	56.44	.0274	.701	25.6
.003575	.236	66.	.0506	.9973	19.71
.00906	.435	48.	.0817	1.147	14.04
.01289	.5143	39.9	.1097	1.234	12.5
.01969	.6773	34.4	.2118	1.3765	6.5

TABLE II.

T	$E = .236$		$E = .4349$		$E = .701$		$E = 1.147$	
	P	r	P	r	P	r	P	r
.00006			.2176	24.	.3570	12.6		
.00018	.0898	40.6	.136	33.			.2973	10.4
.000295					.2352	17.		
.00033	.0512	5.7	.0671	40.6				
.00053			.0634	41.	.1753		.1911	11.7
.00073			.0544	42.		19.2		
.00090	.285	58.						
.00100			.0364	44.				
.00680	.0215	60.	.0309	44.6	.1366	20.6	.1666	12.
.01400	.018	61.	.0245	45.3			.1503	12.2
.0800			.0182	46.				

week during preliminary tests, and the experiment itself continued through another week. At the end of this interval the plates were

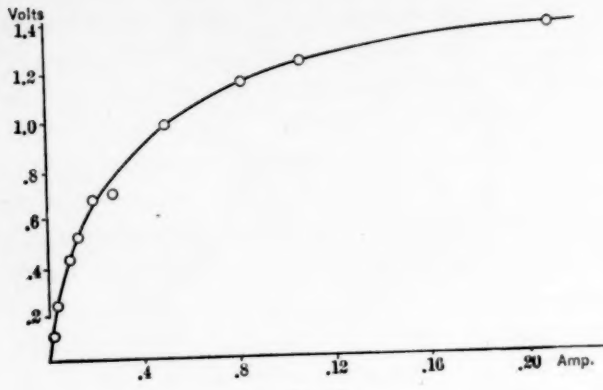


Fig. 2.

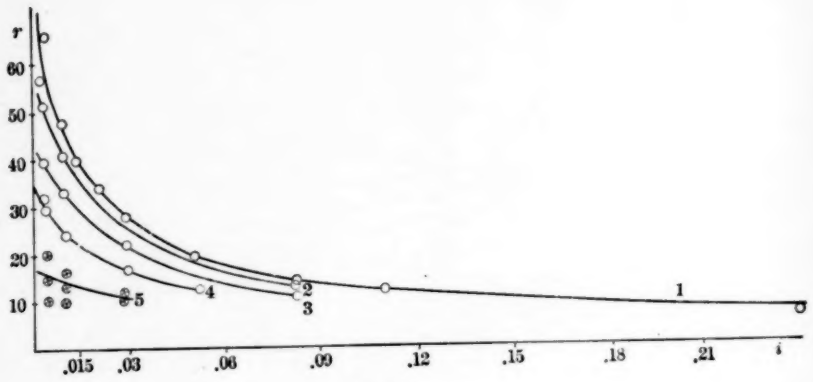


Fig 3

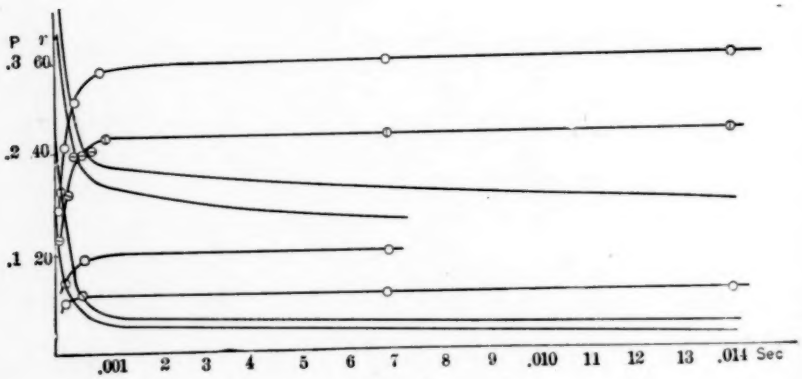


Fig. 4.

colored a dark green, but the uniformity of the readings indicate that the plates did not change throughout the experiment. The test of the resistance by the Kohlrausch method gave a value of thirty-three ohms, but the capacity effect was so large that the adjustment could not be made closer than about five per cent. The measurement of resistance by direct current indicated values much smaller than this. Later it appears that the direct current flowing only for a brief interval had an apparent temporary cleaning action upon the electrodes. The old plates were then thoroughly cleaned and immersed in the same vessel in another solution from the same stock bottle. The resistance now obtained by Kohlrausch's method was 3.9 ohms. The areas of each of these plates was 60 sq. cm. The primary circuit was closed approximately .08 of a second. Fig. 2 shows the relation of the polarizing electromotive force to the current. The ratio of these two quantities is the apparent resistance. In Fig. 3 resistances are plotted as ordinates and currents as abscissæ. The upper curve gives the apparent resistances. The other curves give resistances after allowing for the polarization measured after regular time intervals. The curve nearest the apparent resistance shows the resistance obtained when the polarization is measured, .00033 second after the removal of the polarizing electromotive force. This is the usual character of the curve obtained by the various experimenters, though the time period usually is several times longer than .00033 second. As this time interval is shortened, the resistances diminish and show less variation with changes of current. Some observations are shown on this curve for an unknown but very short time period, the data for which are not recorded. For more minute time period one could expect the curve for resistance to run parallel with the axis of abscissæ, *i. e.*, the resistance would be the same for all currents and probably the same as the electrolytic resistance. Fig. 4 gives the relation of polarization and of resistance to time for four different electromotive forces. As is apparent, the great changes all occur in the first one-thousandth of a second. There are also greater changes for small than for large electromotive forces.

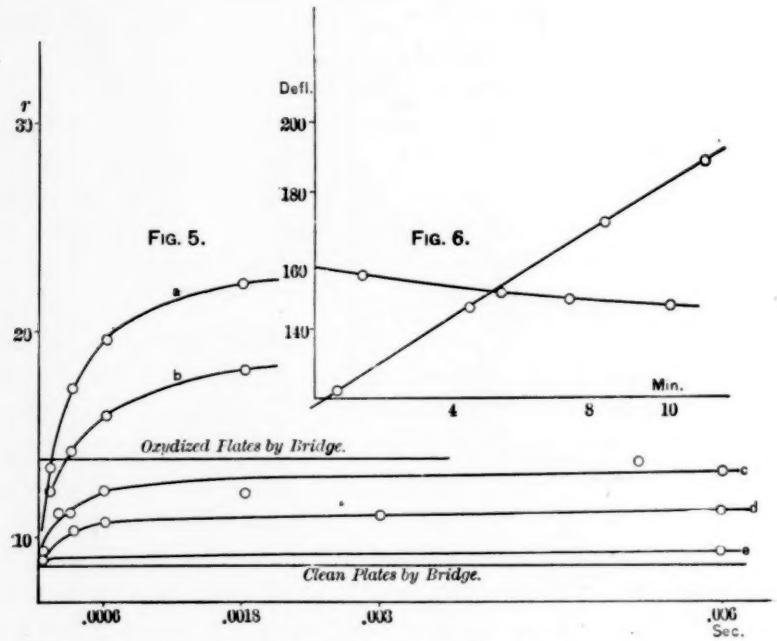
Table III. gives some results obtained with small copper plates with one-tenth the area of the previous plates. The electrolytic

resistance of the solution by the Kohlrausch method was 8.5 ohms when the plates were clean and 13.6 ± 0.4 when they had stood

TABLE III.

$T \times 10^5$	$R_v = 73.5$ $i = .00186$ $E = .1367$		$R_v = 51.4$ $i = .00535$ $E = .3576$	
	P	r	P	r
12	.1098	13.36	.209	12.36
30	.1049	17.10	.200	14.02
60	.1004	19.51	.190	15.86
180	.0955	22.14	.1825	17.91

in the solution for some time. In this experiment the plates had stood in the solution twenty-four hours. The results are shown graphically in curves *a* and *b*, Fig. 5.



Tables IV. and V. give results for plates of the same size as in Table III. The data, however, are for plates when first placed in the solution. Oxidation occurs so rapidly that it was impossible at first to get any consistent results. The resistance increased con-

tinually as indicated both by the potential and the bridge methods. In twenty minutes the plates would be appreciably colored and had to be recleaned. The fresh prepared plates would give new

TABLE IV.

R_v	i	E
16.11	.00050	.0081
15.2	.00193	.0283
13.12	.00454	.0596
12.55	.00587	.0737
12.6	.04580	.6771
11.	.08980	.9878

TABLE V.

	R_v= 15.42 i = .00193 E = .0298		R_v= 12.71 i = .00587 E = .0745		R_v= 11. i = .0898 E = .9878	
	P	r	P	r	P	r
8	.0119	9.26	.0217	9.02		
18	.0083	11.11				8.8
30	.0083	11.11	.0147	10.21		
60	.0062	12.22	.0123	10.61		
180	.0064	12.1				
300			.0102	10.98		
600	.0054	13.15	.0087	11.22		9.2

values so different from the others that the difference could not be accounted for upon the ground of experimental error. Measurements were then taken at stated intervals and when these were exterpolated to zero time they became acceptably uniform. This indicated that the plates, if always cleaned with care and by the same method, would give almost the same resistance when first placed in the solution. The effects of oxidation increased the resistance fairly uniformly from this instant, but the effects were no two times alike and hence the above irregularities. It therefore became necessary to clean the plates for each determination. The circuits were then adjusted for the measurement of polarization, the plates dropped into the solution and about two seconds later the pendulum allowed to swing. The adjustments were then

made for drop over voltameter, V, or over resistance, R of 30 ohms in series with V, and after a definite time interval the pendulum was again dropped. An example of the observations for fall of potential over V and R is given in Table VI. and shown graphically

TABLE VI.

T min.	Drop over 30 ohms.	Drop over V.
.75		122.2
1.5	27.6	147.
4.5		
5.5	270.	
7.5	268.4	
8.5		171.6
10.5	267.6	
11.5		189.

in Fig. 6. The intercept of these curves on the y axis is taken for the true drop over R and V. The plates were then recleaned and the keys set for another period of polarization and observations repeated. Any slight variation in voltameter resistance gives a corresponding variation in applied potential, polarization and current. The mean of these values is recorded in Table IV. A more complete set of observations than given in Table V. is desirable, but, owing to necessity of cleaning the plates, to obtain such a set is an extremely tedious task. The data for resistance in Table V. are shown graphically in curves c, d and e of Fig. 5. Curves a and b arose from polarizing electromotive forces lying between the potentials causing the polarization d and e and that is where a and b should be expected to fall if the plates used in Table III. had been clean instead of being oxidized. We may therefore say that a large part of the effect in a and b arises from a change in the condition of the surface of the electrodes and consequent changes in adjacent solution and that the effect in c, d and e arises largely from the true concentration series.

In the next experiment the polarizing electromotive force varied from .0003 second to .1637 second. The copper plates were the same as in the previous experiment. The resistance external to voltameter was constant. As the time of the applied electromotive

force increased the polarization also increased and there was, therefore, an increase in the terminal electromotive force. This occasions an increase in the apparent resistance of the voltameter and a corresponding decrease in the current. These results are shown in Table VII. For each polarizing period the polarization was measured at different intervals. The results obtained are given in Table VIII. Fig. 7 shows the results graphically. T_v represents the

TABLE VII.

T	i	Rv	E
.0003	.001954	16.84	.0329
.0030	.001927	23.12	.0430
0240	.001914	38.10	.0729
0634	.001895	48.64	.0923
.1637	.001885	54.13	.1010

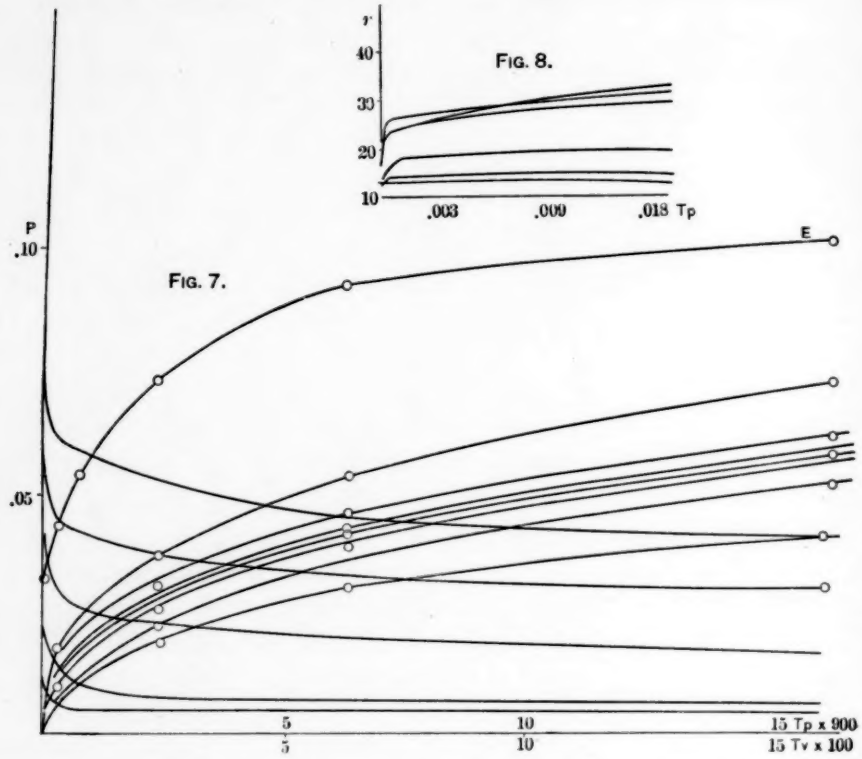
TABLE VIII.

T_P	Tv = .0003		Tv = .003		Tv = .024		Tv = .0634		Tv = .1637	
	P	r	P	r	P	r	P	r	P	r
.00012	.0077	12.90	.0181	13.71	.0367	18.94	.0529	20.44	.0725	15.66
.00030	.0067	13.24	.0152	15.21	.0302	22.32	.0453	24.46	.0611	21.71
.00060	.0058	13.89	.0123	16.72	.0274	23.81	.0425	25.96		
.00120	.0058	13.89	.0093	18.28	.0257	24.71	.0414	26.51	.0575	23.63
.00300	.0058	13.89	.0078	19.09	.0257	24.71	.0409	26.78	.0518	25.56
.00600	.0056	14.02			.0221	26.60	.0388	27.88	.0518	26.66
.01800	.0043	14.69	.0075	19.54	.0166	29.49	.0300	32.52	.0404	32.70

r by alternating current = 13.6.

time the voltameter was polarized and T_p the time elapsing between breaking the circuit and measuring of the polarization. The ordinates represent electromotive forces for both sets of curves. The curves with ordinates increasing with abscissæ each correspond to a constant T_p and variable T_v. For the remaining curves T_v is constant and T_p variable. The abscissæ are nine times larger for variable T_p than for variable T_v. The former show the decrease of polarization when the polarizing electromotive force is removed; the latter show the increase of polarization when the current is applied to the voltameter.

Fig. 8 shows the value of r as ordinates and T_p as abscissæ. They show no regularity as to each other. With more accurate measurements they possibly would be found to intersect the y axis at the point corresponding to the electrolytic resistance.



These observations show clearly the rapid formation of the polarization and that polarization arises for even minute current thrusts and for alternating currents. At first the current produces a change at the contact of solution and electrodes. It then effects deeper layers of the solution and the effect at the contacts of electrodes and solution increases, but less rapidly. This gives rise to further, but smaller increases in polarization. Diffusion diminishes the increase in polarization and after a time there is a constant potential between the terminals of the voltameter or equilibrium between the diffusion process and the increase in potential. On open circuit the diffusion still continues and affects most rapidly the strata nearest

the electrodes were in many cases a reaction occurs. Therefore at the electrode surfaces the changes are greatest. When all the strata affected by the concentration changes are extremely thin, then the polarization will decrease very rapidly. This effect is shown in the polarization curves and the rapid changes of the apparent resistance when the applied current was very small.

When the currents become larger there are concentration changes farther from the electrodes, and the greater the distance from the electrodes to the outlying concentration changes the less P will diminish in a short interval. Therefore, the values obtained for the true resistance when large currents are used, provided they are not so large as to complicate the phenomena, correspond more nearly to values obtained with alternating currents. Since the circuits were not closed long enough to establish a maximum of polarization, one might anticipate that the same effect would be produced by increasing the length of time, the polarizing electromotive force was applied. In Table VIII. it is seen the polarization does increase as T_v increases and that the residual polarization is more persistent on open circuit for the larger values of T_v. However, when the current is applied for a longer time the curve for apparent resistance does not become more nearly horizontal but becomes more bent. This would indicate that as the polarizing electromotive force increases and the current decreases the part of the polarization at the contact of the electrode and the solution increases more rapidly than the part farther removed from the electrodes. Therefore, there is a greater change in polarization for about the same current, when the time for the polarizing electromotive force is increased. This point, however, needs further investigation. The intercept on the y axis of the E curve divided by the current gives the electrolytic resistance as measured by the alternating current. The polarization curves with varying T_v all pass through the origin. The value of the polarization T_v for $T_p = 0$ would be difficult to exterpolate. These polarization curves approach nearer the E curve as time increases and makes r apparently increase and later decrease with increasing time. This may be due to some inaccuracy in the measurements, but it is not altogether impossible since the polarization is very persistent when the time is longer as indicated by the

curves where T_p is variable. The curves plotted with the last factor as variable indicate that there is a considerable part of the polarization very persistent. There is a part also which diminishes very rapidly, and which, by ordinary methods, has not been observed. The former probably concerns the strata of the concentration series more remote from the electrodes. The latter probably arises from the outer strata of the concentration series adjoining the electrode, where in the case of the oxidized plates reaction at once sets in. In case of clean copper plates this action would be scarcely noticeable, and therefore the slight variation of the *c*, *d* and *e* curves, Fig. 5. The curves and the data also indicate that if the true maximum value of P could be measured, the simple formula in ordinary use would give nearly the true resistance of the cell for all currents. That it should give the true electrolytic resistance when the measurements are very accurately measured should not be expected under all conditions. For concentration changes mean layers of different conductivity from the body of the solution. The heating of the electrodes may not be altogether negligible, particularly if either electrode is small and current density large. Gaseous products, and a change in the aggregation of the electrode surface also produce changes, since the current must either pass around or through them. Such phenomena as these together with the unmeasured part of the polarization, constitute what has been designated as a "transition resistance." It is always desirable to have as few complications as possible.

The time periods here used are so short one can estimate the polarization which would arise when an alternating current of slow frequency, like that used in electrolytic measurements, is applied. Let us assume a current alternating five hundred times to the second, having the same current strength as used in this experiment. Assume the average electromotive force to be seven-tenths of the maximum. Then the maximum would be reached in .001 second and the polarization would therefore be about .01 volt.

It is the author's intention to farther apply this method to thoroughly oxidized plates, and to clean electrodes, if oxidation of the latter can be prevented. Formaldehyde is recommended by some to prevent oxidation and will be tried. Farther tests for

capacity; and for other solutions and other electrodes are desirable. To diminish the time period much more will not be possible since the shortest time already is near the limit of the time for the condenser to acquire full charge. However, a change in the apparatus so as to obtain greater accuracy in the measurement of the time is very desirable. The method is also applicable to measurement of internal resistance and polarization in primary batteries.

UNIVERSITY OF NEBRASKA, June, 1899.

A NEW METHOD OF COMPARING TWO SELF-INDUCTANCES.

HUBERT V. CARPENTER.

DURING a careful study of the more recent methods of measuring and comparing inductance and capacity I have developed the following method of comparing two self-inductances which gives very accurate results. Its freedom from the effects of other inductances permits its use for the comparison of very small values, and of values which are very different.

If l_1 and l_2 are the two coils whose coefficients of self-inductance, L_1 and L_2, are to be compared, connect the two in parallel having in series with one of them a variable non-inductive resistance. Apply to their common terminals an harmonic electromotive force. This will cause currents to flow through the coils which will lag behind the impressed electromotive force by the phase angles:

$$\varphi_1 = \tan^{-1}\frac{\omega L_1}{R_1}, \text{ and } \varphi_2 = \tan^{-1}\frac{\omega L_2}{R_2},$$

in which R_1 and R_2 are the total resistances of the two parallel branches including the coils l_1 and l_2, the variable resistance in one branch, and the connecting wires; while ω is equal to 2π times the frequency of the alternating current or $2\pi n$. This shows that when the two ratios $\frac{\omega L_1}{R_1}$ and $\frac{\omega L_2}{R_2}$ are equal the two currents in the parallel branch will be in the same phase. If, then, we can by any means produce and detect this condition of phase equality, we get at once the relation between the two inductances. It is simply, $\frac{\omega L_1}{R_1} = \frac{\omega L_2}{R_2}$, or $\frac{L_1}{L_2} = \frac{R_1}{R_2}$.

The two primary currents can be brought into phase equality by varying the non-inductive resistance in one of the branches. This condition can be detected in the following manner: Introduce a sec-

ondary circuit which is made up of two equal coils, s_1 and s_2, of many turns, and a telephone, all connected in series. Place coil s_1 close and parallel to l_1, coil s_2 close to l_2. The currents in l_1 and l_2 then induce electromotive forces in s_1 and s_2 respectively. The geometric sum of these two secondary electromotive forces will cause a current through the telephone. Suppose, however, the coils s_1 and s_2 to be so connected that when the currents in l_1 and l_2 are in phase with each other, the induced electromotive forces, Es_1 and Es_2 will be opposed to each other in the secondary circuit. The resultant secondary electromotive force, Es, will then be equal to $Es_1 - Es_2$; or, if these be made equal, will be zero. This condition will be detected by the silence of the telephone.

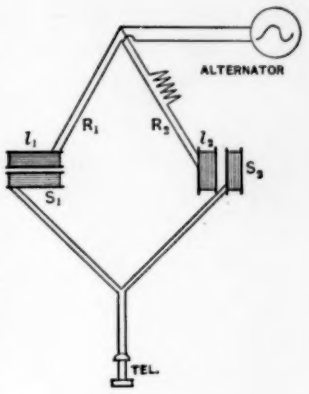


Fig. 1.

This silence indicates two things: 1st, that the two primary currents, Cl_1 and Cl_2, are in the same phase; 2d, that $Es_1 = Es_2$. Since the first condition is true, when $\frac{\omega L_1}{R_1} = \frac{\omega L_2}{R_2}$, it can be made true by varying R_1 or R_2. The second condition may be secured by varying the relative position of one of the secondary coils, S_2 for instance, with respect to its primary. This will change the electromotive force induced in it, and if it be moved back and forth until a position is found where the hum in the telephone is a minimum or zero we know that it is so placed that $Es_1 = Es_2$. If this adjustment does not give silence but only a minimum sound in the telephone it shows us that the two electromotive forces, Es_1 and Es_2, are not in exact opposition. This means that the two primary currents, Cl_1 and Cl_2, are not in the same phase and so R_1 or R_2 must be changed. After the value of R_1 or R_2, and the position of S_2 have been adjusted so as to give silence in the receiver, it is only necessary to know $\frac{R_1}{R_2}$ to get $\frac{L_1}{L_2}$.

The proper position of S_2 can be found so quickly that the work

of obtaining a balance is but little more tedious than in the Wheatstone's bridge. When using the telephone as a current detector, however, there is no means of knowing which way R_1 or R_2 must be varied to reduce the phase difference between Cl_1 and Cl_2. This corresponds to the same weakness of the telephone when used in the Wheatstone's bridge.

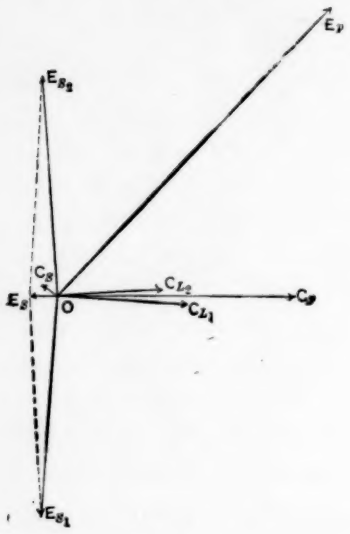


Fig. 2.

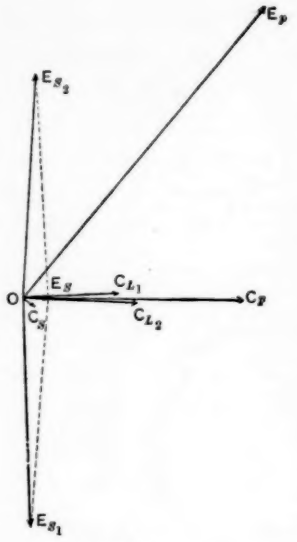


Fig. 3.

The phase relations of the currents and electromotive forces just discussed may be seen upon the two phase diagrams, Fig. 2 and Fig. 3. In these diagrams the vectors represent by their length the maximum values of the currents and electromotive forces, while their directions indicate their respective phases. The instantaneous values of all the quantities at any instant is given by the lengths of the projections of their respective vectors upon a moving vector which rotates in the clockwise direction through the entire 360° with each cycle of the current. According to this the secondary electromotive force, Es_1, reaches its maximum one quarter of a cycle later than the current Cl_1 which induces it, as it should. The vector Es_2 is shown reversed or 180° out of its true position. This corresponds to the physical condition of being connected in series with Es_1 in such a way that they oppose each other. As may be

seen from the two diagrams, the resultant secondary electromotive force may have any phase whatever and varies with every change in the value of R_1 or R_2, or in the position of the coil S_2.

Careful tests were made of the method in this form. With a fairly good telephone, whose resistance was much too low for the greatest sensitiveness, successive comparisons could be made to agree within one-half of 1 per cent. without difficulty.

In order that these results may not only agree among themselves, but may be correct, the apparatus must be so arranged that there shall be no appreciable inductive effects except those which are accounted for. To do this the two coils under comparison, l_1 and l_2, which carry fairly large currents, must be so placed as to prevent mutual inductance between them. This was accomplished by placing them about two meters apart and in mutually perpendicular planes. Tests were made which showed this to be effectual. The lead wires from the source of the primary current to the point where the circuit divided were made up of twisted lamp cord and were kept as far as possible from the coils. The leads to each coil from this dividing point were placed about one centimeter apart and crossed over at frequent intervals. In comparing coils of low inductance it would be best to use lamp cord here also, as the effect of capacity would be very small compared with the inductive effect of the large current.

Any current which flows in the secondary circuit will, of course, react upon the primary currents, but when a balance is reached the secondary current disappears, and with it its disturbing influence. The secondary electromotive forces remain, however, and, together with the distributed capacity in the secondary coils, will set up a small local current in each coil. The amount of this hidden current is hard to estimate. It depends upon the capacity of the coil, which we can do no more than estimate, and its value is different at every point along the conductor. It can easily be shown to be very small however. Two pairs of coils were used at different times as secondaries, one pair of 300 turns each of No. 25 B. & S. wire and about 30 cm. in diameter; the other pair of 3000 turns each of No. 30 wire and about 10 cm. in diameter. The latter were made up of 50 layers of 60 turns each. If we were to con-

sider each layer as a plate of a condenser with alternate plates connected together the calculated capacity would be about .012 microfarad. In the coil, however, the difference in potential between layers is only about one-fiftieth of the potential across the terminals of the coil so the effective capacity will be something like .0002 microfarad. Suppose the total electromotive force induced in the coil to be as great as 50 volts or one volt per layer, and assume that the resistance and self-induction of the coil does not prevent it from becoming fully charged at each impulse of the current. Then, if the current in the primary passes through sixty cycles per second the local current in each secondary coil will be :

$$I = \frac{Q}{t} = \frac{EC}{t} = 120 \times 1. \times .0000000002 \text{ or about } 24. \times 10^{-9}$$

ampères. If the coils of 300 turns each were used the local current would be much less than this. If this estimate be correct it shows that the reactive effect of these local currents upon the primary coils is entirely negligible. Further, it is easily seen that both primary coils are affected in the same way so that if the primaries were equal the error would be zero. Tests were made with both pairs of secondary coils upon the same primaries and the fact that no difference in the results due to the change if secondaries could be detected gives experimental evidence that the effect of the local currents is not measurable.

The next step in the development of the method was to replace the telephone by the electro-dynamometer. This required considerable change in details, but the principle remains the same, and the results showed a marked improvement in accuracy. The ordinary dynamometer having both coils connected in series is not sensitive to small currents, so a special instrument was used, in which the fixed coil was made up of 600 turns of No. 25 wire, and the suspended coil of 380 turns of No. 36. The latter was suspended by a phosphor bronze strip of about three circular mils cross-section and ten inches in length. This suspension strip and a similar strip in a loose spiral below the coil formed the connecting wires to the suspended coil. These strips would carry about .01 ampère for a short time, while the fixed coils of the instrument would safely carry 1,000 times as great a current for the same length of time. No

metal was used in the instrument near the coils except the conductors themselves, because it was designed for use in testing Professor Rowland's phase angle methods. In the new method it is used only to detect small currents, and for this could be improved by adding more turns to the suspended coil and an iron core to the fixed coils.

The small secondary current, which is to be made zero, is passed through the suspended coil, and the heavy primary current through the fixed coils of the dynamometer. In this way the instrument was made sensitive to currents of about $10.^{-8}$ ampères. The direction of the deflection of the suspended coil reverses when the phase of the current in either coil is changed by 180°, so the instrument will give an indication concerning which way the resistances of the primary branches must be changed to bring the balance nearer. This is another advantage over the telephone. One peculiarity of the dynamometer must be remembered, however. If the two currents in the two coils differ by 90° in phase there will be no deflection. It becomes necessary, then, to distinguish between a zero dedeflection due to zero secondary current and a zero deflection due to 90° phase difference between the primary and secondary currents. The phase relations between these two currents may be seen in the phase diagrams, Figs. 2 and 3. We see that without some other arrangement it is impossible to tell whether the dynamometer is indicating zero secondary current or 90° phase difference. This difficulty was met in the following way: A non-inductive shunt, R_n, was connected across the terminals of the fixed coils of the dynamometer, with a key in its circuit so that it could be cut out at will. When this is connected the main primary current is divided between the two paths, and since the coils have considerable inductance while the shunt has not, the current in the fixed coils will lag behind the total primary current by a considerable angle. The currents and electromotive forces of the coils, l_1, l_2, s_1, s_2, do not change their relative phases in any way when this shunt connection is made, so the result is that the difference in phase between the two dynamometer currents is altered. If it was 90° before the shunt was connected, it must be something more or less afterwards, and unless the secondary current is zero, there will be a deflection of the suspended

coil. This result could have been attained easily in other ways, but this seemed the simplest and proved very satisfactory.

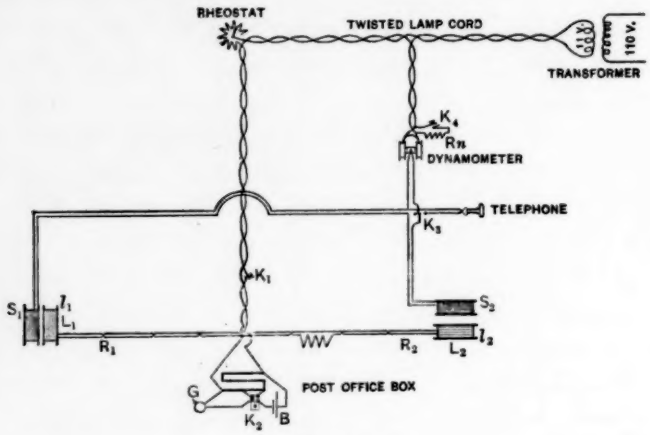


Fig. 4.

The connections as finally used are shown in the large diagram, Fig. 4.

The theory so far given is dependent upon the assumption that the electromotive force impressed upon the primary circuit is an harmonic function of the time. Since, however, the ratio of inductance to resistance in the two branches of the primary circuit are made the same at the final balance, it is true that, no matter what electromotive force is applied to their common terminals the currents in the two branches will rise and fall together. The two secondary electromotive forces will then have the same wave form and phase, and can be made to neutralize each other completely. This was shown to be true experimentally by the fact that when a balance had been obtained by the use of the dynamometer and an alternating current, no sound would be heard in the telephone when an interrupted current was used. This extreme test is certainly ample verification. The question of the source of the electromotive force is now an easy one, for any alternating current of any frequency, wave form, or degree of unsteadiness may be used. The tests given herewith were made with current from a two-phase, 60-cycle, Westinghouse generator. No tests of the wave form were made but the

makers state that it differs from a sine curve by less than five per cent. at every point.

Let us consider the conditions under which the method will give the best results. In short, what is the relation between L_1 and R_1, or between L_2 and R_2, for maximum sensitiveness? The resultant electromotive force in the secondary circuit, E_s, also the secondary current, C_s, upon which the observations depend, varies, for small differences, with the angle of phase difference between Cl_1 and Cl_2. For maximum sensitiveness a given small change in the value of R_1 or R_2 should produce a maximum change in this angle, θ, of phase difference; or, the rate of change of R_1 with respect to θ should be a minimum. θ is equal to $\varphi_1 - \varphi_2$, where φ_1 and φ_2 are the phase angles by which the currents Cl_1 and Cl_2 lag behind the primary electromotive force, and to produce a change in θ we change either φ_1 or φ_2 while the other is unaltered. It follows then that a given change in φ_1 will produce an equal change in θ. This being true we can say that, for maximum sensitiveness, the rate of change of R_1 with respect to φ_1 must be a minimum.

The relation between R_1 and φ_1 is,

$$\tan \varphi_1 = \frac{\omega L_1}{R_1}$$

or,

$$R_1 = \omega L_1 \cot \varphi_1$$

The rate of change of R_1 with respect to φ_1 is then:

$$\frac{dR_1}{d\varphi_1} = -\omega L_1 csc^2 \varphi_1.$$

This rate will be a minimum when:

$$\frac{d^2 R_1}{d\varphi_1^2} = 0, \text{ or when } 2\omega L \cot \varphi_1 csc^2 \varphi_1 = 0.$$

Substituting for the trigonometric functions their values in terms of ωL_1 and R_1 we have,

$$2 \cdot \omega L_1 \cdot \frac{R_1}{\omega L_1} \cdot \frac{R_1^2 + \omega^2 L_1^2}{\omega^2 L_1^2} = 0.$$

Since R_1 is the only variable included in the expression we see that the most accurate results will be attainable when the resistances R_1 and R_2 are as near zero as possible. In other words, the greater

the angle by which the primary currents lag behind the electromotive force the greater will be the accuracy of the results. The formulæ also show that the sensitiveness when the primary currents lag 45° is one-half of the theoretical maximum. This is attainable in the case of most ordinary coils with a frequency of sixty.

As may be seen from Fig. 4 the telephone and dynamometer were connected so that either could be used in the secondary circuit. In practice the telephone was used in the preliminary adjustments and the dynamometer for obtaining the balance. The postoffice box was connected to the central posts where the leads to the primary coils were joined and as soon as an adjustment was completed a lead wire of one coil, say l_1, was disconnected and the resistance of l_2 measured; then l_2 was disconnected and l_1 reconnected and its resistance measured. In this way R_1 and R_2 were determined before any appreciable change could occur for care was always taken to avoid heating the coils. The postoffice box used was the best in the laboratory and can be depended upon to give results within $\frac{1}{10}$ % of the correct values while the ratios given by it are considerably more accurate because the two resistances were measured under the same conditions and as far as possible the same coils in the box were used. The resistances given below have all been corrected for the resistance of the leads from the postoffice box to the common terminals of the coils.

The inductances of five different coils were compared. The coils will be designated, l_1, l_2, l_3, l_4, and l_5, with inductances, L_1, L_2, L_3, L_4 and L_5, respectively. Coils l_1 and l_2 were wound in regular layers in a rectangular groove cut in a disk of plaster of paris, and their dimensions accurately determined so that their inductances could be calculated. Coil l_1 was made up of 70 turns in 7 layers of No. 20, B. & S. gauge, single silk-covered copper wire. Its mean diameter was 38.605 cm. Calculated inductance, $L_1 = .0049046$ henry.

Coil l_2 was made up of 8 turns in 2 layers of No. 15 B. & S gauge, double cotton-covered copper wire. Its mean diameter was 38.681 cm. Calculated inductance, $L_2 = .0000726$ henry. These inductances were calculated by the formula due to Maxwell and Stefan.[1] The value calculated for L_1 was assumed to be correct

[1] Stefan, Wied. Ann., V. 22, p. 107, Apr., 1884.

and made the basis for finding the values of the others from the ratios. Coils l_3, l_4 and l_5 were of unknown dimensions and wound on wooden frames.

Data. Table I.

COMPARISON OF COILS l_1 AND l_2.

R_1	R_2	L_2	
3.1574	27.865	.043285	
3.1198	27.559	.043324	
3.1104	27.479	.043327	
3.0940	27.363	.043375	
3.0690	27.115	.043330	
		Average =	.043328
	Probable error = .000010	=	.02 + per cent.

Table II.

COMPARISON OF COILS l_1 AND l_4.

R	R_4	L_4	
2.9785	30.828	.050762	
2.9900	30.985	.050825	
3.0212	31.209	.050662	
3.0380	31.459	.050790	
2.9911	30.978	.050793	
2.9068	30.038	.050682	
2.9542	30.594	.050790	
2.9357	30.427	.050830	
		Average =	.050767
	Probable error = .000014	=	.03 — per cent.

Table III.

COMPARISON OF COILS l_3 AND l_4.

R_3	R_4	L_4	
26.855	31.450	.050741	
26.905	31.547	.050802	
26.897	31.540	.050808	
26.873	31.427	.050665	
26.800	31.406	.050774	
27.023	31.647	.050740	
26.915	31.513	.050730	
26.950	31.520	.050675	
		Average =	.0507445
	Probable error = .000012	=	.02 + per cent.

$L_4 = \frac{R_4}{R_3} L_3$ where $L_3 = .043328$

TABLE IV.

COMPARISON OF COILS l_1 AND l_5.

R_1	R_5	L_5	
3.1641	.26870	.00041650	
3.1800	.26911	.00041510	
3.1905	.27033	.00041555	
3.1847	.27009	.00041593	
3.1838	.26996	.00041585	
3.1775	.27013	.00041695	
3.2108	.27266	.00041650	
		Average =	.00041605
	Probable error = .00000016	=	.04 — per cent.

TABLE V.

COMPARISON OF COILS l_2 AND l_5.

R_2	R_5	L_2	
.76554	.13177	.000071615	
.76513	.13182	.000071680	
.76579	.13185	.000071628	
		Average =	.000071641

The values given for L_2 in Table V. are based on the value of L_5 found in Table IV. This depends upon the calculated value of L_1, so that in the approximate agreement between the above value and the calculated value of L_2 we have an inaccurate check upon the correctness of both. Table III. gives a direct check upon the correctness of the ratios $\frac{L_1}{L_3}$, $\frac{L_1}{L_4}$ and $\frac{L_3}{L_4}$. From the data there given the ratio of L_3 to L_4 is obtained, and the values of L_4 given are based upon the value of L_3 obtained in Table I. The two values L_4 differ by 0.05%, but this represents the combined error in three independent sets of observations.

The method here outlined might also be applied to the comparison of two capacities by connecting one in series with each primary coil l_1 and l_2. The results, however, would require correction for the inductances L_1 and L_2, and, further, unless the two condensers were exactly similar in construction, the result would be in error, because of the unequal rates of absorption.

PHYSICAL LABORATORY, UNIVERSITY OF ILLINOIS, April, 1899.

NEW BOOKS.

Lehrbuch der Experimentalphysik. Vierter Band; *Die Lehre von der Strahlung.* By ADOLPH WÜLLNER. Leipzig: B. J. Teubner, 1899.

The fourth volume of the fifth edition of this standard treatise deals with radiation. Its new title differs from that of the corresponding volume of the earlier editions, which was entitled *Die Lehre vom Lichte.* The work has grown in the attempt to incorporate the results of the most important newer investigations from 624 pages in the third edition of 1875, to 1,042 pages in the edition of 1899.

This volume shows indeed a greater amount of new material than any of the other volumes of this edition, and while the plan of the work remains unchanged, one is pleased to recognize the introduction of full and very satisfactory descriptions of such investigations as that of Wiener, on "Standing Light Waves," of Kerr on "Electro-optic Double Refraction," of Drude and of Rubens on "Metallic Refraction," etc.

The most important portions of Zeemann's work on the influence of "Magnetization and the Emission of Light," appeared too late to find a place in this edition, but his earlier papers of 1897 are briefly described.

E. L. N.

Ostwald's *Klassiker der Exakten Wissenschaften.* No. 99, *Ueber die Bewegende Kraft der Wärme,* by R. CLAUSIUS. No. 100, *Abhandlungen über Emission und Absorption,* by G. KIRCHHOFF. Edited by Professor MAX PLANCK. Leipzig: Wilhelm Englemann, 1898.

With these two volumes, this well-known series of German scientific-reprints reaches its 100th number. Of the two memoirs selected by the editor, Professor Planck, for reprinting, the papers of Kirchhoff were perhaps more in need of being brought into convenient and accessible form than the well-known work of Clausius, on the "Motive Power of Heat." Both indeed are easily accessible in German, the latter having been reproduced in the first edition of Clausius's treatise while Kirch' hoff's memoirs are to be found not only in the Transactions of the Berlin Academy of Science, a set which is not always at one's disposal, but likewise in his volume of *Gesammelten Abhandlungen.* Of Clausius paper there exist at least three literal English versions. The value to

science in these reprints will be found, therefore, rather in the two brief notes of the editor than in the reproduction of the texts themselves.

E. L. N.

Inorganic Chemical Preparations. By FELIX LENGFELD. Pp. xvii+55. New York, The Macmillan Co., 1899. (*Received.*)

Laboratory Manual. Experiments to illustrate the Elementary Principles of Chemistry. By H. W. HILLYER. Pp. vi+200. New York, The Macmillan Co., 1899. (*Received.*)